J. Fluid Mech. (1978), *vol.* 89, *part* 3, *pp.* 433–468
Printed in Great Britain

Unsteady vortical and entropic distortions of potential flows round arbitrary obstacles

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(Received 24 October 1977 and in revised form 10 February 1978)

This paper is concerned with small amplitude vortical and entropic unsteady motions imposed on steady potential flows. Its main purpose is to show that, even in this unsteady compressible and vortical flow, the perturbations in pressure p' and velocity $\mathbf{u}$ can be written as $p' = -\rho_0 D_0\phi/Dt$ and $\mathbf{u} = \nabla\phi + u^{(I)}$ respectively, where D_0/Dt is the convective derivative relative to the mean potential flow, $\mathbf{u}^{(I)}$ is a *known* function of the imposed upstream disturbance and ϕ is a solution to the linear inhomogeneous wave equation

$$\frac{D_0}{Dt}\left(\frac{1}{c_0^2}\frac{D_0\phi}{Dt}\right) - \frac{1}{\rho_0}\nabla.(\rho_0\nabla\phi) = \frac{1}{\rho_0}\nabla.\rho_0\mathbf{u}^{(I)}$$

with a dipole source term $\rho_0^{-1}\nabla.\rho_0\mathbf{u}^{(I)}$ whose strength $\rho_0\mathbf{u}^{(I)}$ is a known function of the imposed upstream distortion field. (Here c_0 and ρ_0 denote the speed of sound and density of the background potential flow.) This equation is used to extend Hunt's (1973) generalization of the 'rapid-distortion' theory of turbulence developed by Batchelor & Proudman (1954) and Ribner & Tucker (1953). These theories predict changes occurring in weakly turbulent flows that are distorted (by solid obstacles and other external influences) in a time short relative to the Lagrangian integral scale.

The theory is applied to the unsteady supersonic flow around a corner and a closed-form analytical solution is obtained. Detailed calculations are carried out to show how the expansion at the corner affects a turbulent incident stream.

1. Introduction

Much of both aerodynamics and hydrodynamics is concerned with high Reynolds number flows produced by solid bodies moving through a fluid at rest or, equivalently, with stationary bodies or obstacles embedded in nearly inviscid flows that have constant velocity and physical properties far upstream. Major portions of these flows have velocity fields that can be expressed as the gradient of a scalar potential.

There have also been many studies of the alterations that are produced when small amplitude (steady or unsteady) upstream distortions are imposed on such flows. Analyses of this type usually fall into two groups. One of these is concerned with airfoils and other bodies that have at least one small transverse dimension and consequently cause only small departures from the uniform upstream flow. The motion produced by the imposed upstream distortion is then effectively decoupled from the steady potential flow about the body and can therefore be calculated as if the body had zero thickness and angle of attack (which for an airfoil could correspond to a flat plate at zero angle of attack and with the same projected area as the airfoil). The entire flow arising from the interaction between the imposed distortion and the obstacle is then the result of a blocking effect caused by the vanishing of the normal velocity at the surface of the body. Such flows are described by equations with constant coefficients and solutions can therefore be found for a wide variety of conditions. There is no need to assume that the flow is incompressible or that the body is two-dimensional. Studies of these flows are relevant to the prediction of gust loading on airfoils and other aerodynamic surfaces and even to aeroacoustic investigations concerned with aircraft engine-fan and compressor noise. The first solution to this type of problem was given by Sears (1941).

The second class of problems is concerned with flows about blunt bodies and other obstacles that produce a non-negligible disturbance to the upstream flow. Such problems lead to equations with variable coefficients unless the flow is assumed to be incompressible and all investigations of these flows have therefore invoked this assumption. The first work on this category of flows is due to Lighthill (1956), who imposed an upstream vorticity field that was independent of time but varied in space. The most general upstream vorticity field consistent with the assumption that it represents a small disturbance of a uniform flow has recently been treated by Hunt (1973), who used his results to generalize the Ribner–Tucker (1953) and Batchelor–Proudman (1954) 'rapid-distortion' theory of turbulence. The extended theory accounts for non-uniform strains and solid-surface blocking effects. Hunt used his approach to analyse the turbulent flow about a two-dimensional circular cylinder.

One purpose of the present paper is to develop a unified approach that can deal with both categories of flows alluded to above. To this end we consider the most general type of disturbance to the uniform incident stream (which includes both entropy and vorticity disturbances) and we require neither that a transverse dimension of the body be small nor that the flow be incompressible. Another purpose of this paper is to use this approach to study a rather general class of flows that does not fall into either of the categories discussed above and which, to our knowledge, cannot be treated by any other method.

The potential flow upstream of a three-dimensional obstacle or a non-lifting two-dimensional obstacle is uniform enough to ensure that the imposed distortion field will act like a small disturbance on a constant velocity mean flow. The character of such disturbances is well understood (Kovásznay 1953). They can be decomposed into distinct acoustic-, vortical- and entropy-type modes, each of which can exist independently of the others. The vortical mode has a divergence-free velocity field and produces no pressure fluctuations. Since the entropy mode is also decoupled from the pressure fluctuations the latter can be produced only by the acoustic mode. But we are not concerned in this paper with the effect of incident acoustic fields and these are eliminated from the discussion.

In § 2 we consider the effect of the most general non-acoustic incident distortion field that can be imposed on the uniform upstream flow. It is then shown that the perturbation velocity $\mathbf{u}$ at any point $\mathbf{x}$ of the resulting unsteady compressible and vortical flow will consist of (i) a part $\mathbf{u}^{(I)} = \mathbf{u}^{(I)}(\mathbf{x}, t)$ that is a *known* function of the imposed upstream distortion field and the mean flow variables and (ii) a part $\nabla\phi$ that is related to the pressure fluctuations p' by $p'/\rho_0 = -D_0\phi/Dt$, where D_0/Dt is the convective derivative based on the mean flow velocity, $\rho_0 = \rho_0(\mathbf{x})$ is the density of the mean flow and t denotes the time. The 'perturbation potential' ϕ can be found by solving the linear inhomogeneous wave equation

$$\frac{D_0}{Dt}\frac{1}{c_0^2}\frac{D_0\phi}{Dt}-\frac{1}{\rho_0}\nabla.(\rho_0\nabla\phi)=\frac{1}{\rho_0}\nabla.\rho_0\mathbf{u}^{(I)} \tag{1.1}$$

with a dipole-type source term $\rho_0^{-1}\nabla.\rho_0\mathbf{u}^{(I)}$ whose strength $\rho_0\mathbf{u}^{(I)}$ is a known function of the imposed upstream distortion field and the basic mean-flow variables. $c_0 = c_0(\mathbf{x})$ denotes the sound speed of the mean flow.

The known portion $\mathbf{u}^{(I)}$ of the perturbation velocity field is linearly related to the imposed upstream distortion by a simple formula (equation (2.33) below) that is the sum of two terms. The first of these represents the effect of the imposed upstream vortical velocity field $\mathbf{u}_\infty$ while the second represents the vortical velocity generated by the interaction between the imposed upstream entropy fluctuations and the steady potential flow. This formula may be the most important result of the paper.

At upstream infinity ρ_0 becomes constant and $\mathbf{u}^{(I)}$ approaches the imposed vortical velocity field $\mathbf{u}_\infty$. Then since, as indicated above, the latter quantity has zero divergence, the source term in (1.1) will vanish at upstream infinity and the outgoing-wave solution ϕ will therefore approach zero in this region. But the portion $\rho_0\mathbf{u}^{(I)}$ of the momentum perturbation (which becomes equal to $\rho_0\mathbf{u}_\infty$ far upstream) is distorted by the steady potential flow as the latter convects it towards the body surface. This destroys the initial divergence-free property of $\rho_0\mathbf{u}^{(I)}$ and thereby produces a non-zero source term which causes the solution ϕ of (1.1) to be non-zero at finite distances from the obstacle. In this way the distortion effect is able to produce a potential velocity field $\nabla\phi$ with its attendant pressure fluctuations $-\rho_0 D_0\phi/Dt$. If the mean flow were entirely uniform, i.e. if no distortion occurred, the imposed upstream vortical velocity and entropy fluctuation would, as indicated above, satisfy their governing equations without producing any pressure fluctuations. The latter can therefore be attributed to the distortion effect described above.

But pressure fluctuations can also be produced by the boundary condition on the surface of the obstacle. For rigid bodies the normal component of $\mathbf{u} = \mathbf{u}^{(I)}+\nabla\phi$ must vanish on this surface and, since $\mathbf{u}^{(I)}$ will in general not equal zero there, $\nabla\phi$ will also be non-zero. Then ϕ will not vanish in the flow and pressure fluctuations will again be produced.

The effect of airfoils and other two-dimensional lifting surfaces is felt so far upstream in subsonic flows that the imposed upstream disturbance field cannot be assumed to be the same as it would be in a uniform flow (Goldstein & Atassi 1976). However, an appropriate form for the upstream distortion field is deduced in § 2.5 and it is then shown that the theory described above will, with only slight modification, also apply to flows of this type.

In § 3 we use (1.1) to study the unsteady compressible flow over a two-dimensional obstacle whose transverse dimensions are non-negligible. This flow does not fall into either of the categories described above. In order to emphasize the compressibility effects, we suppose that the flow is supersonic and, in order to treat a situation of a general type that is still simple enough to lead to closed-form solutions, we take the mean flow to be a Prandtl–Meyer expansion around a corner. The wave equation (1.1) has variable coefficients for this flow and cannot be solved by separation of variables. However, we can, as a result of a rather remarkable set of circumstances, reduce it to a first-order linear partial differential equation by taking its Laplace transform. The latter equation is then solved in the usual way by the method of characteristics.

In § 4 we use the solution to calculate the turbulence velocity correlations in the expansion fan at the leading edge of a wedge at a large angle of attack to the mean flow. With the recent advances in laser-Doppler velocimetry it is now possible to measure such correlations in supersonic flows. The unsteady motion is assumed to result from a specified upstream turbulence field. The conditions under which the present model describes real turbulent flows are roughly those given by Hunt (1973) with some relatively minor restrictions on the Mach numbers, which are discussed below.

It is frequently argued that there should not be any substantial differences between the turbulence in subsonic and supersonic flows since the turbulent Mach number will be quite small unless the mean-flow Mach number is quite large. But these arguments apply only to parallel shear flows such as those that occur in boundary layers, jets and wakes. The variation in mean velocity is relatively unimportant in such flows and its effect can be largely removed by a Galilean transform which does not alter the equation of motion. The situation is quite different for the type of flow being considered here. These flows usually involve large accelerations and, in order to ensure that the model will remain valid for reasonably strong turbulence, we must assume that the scale of the turbulence is not substantially smaller than the scale on which the mean flow changes.

2. The basic equation

We consider an inviscid non-heat-conducting compressible flow past an obstacle and suppose that the upstream velocity consists of a uniform portion U_∞ on which there is imposed a small amplitude unsteady motion.† Correspondingly, we also suppose that there exist small unsteady perturbations in the otherwise uniform physical properties of the fluid. There are no essential restrictions on the Mach number of the flow. Now the nature of the small amplitude unsteady motion on a uniform flow has been understood for some time (Kovásznay 1953). In such flows the velocity field can be decomposed into the sum of (i) a disturbance (often called a gust) that is purely convected (i.e. frozen in the flow), has zero divergence and is completely decoupled from the fluctuations in pressure or any other thermodynamic property and (ii) an irrotational disturbance that produces no entropy fluctuations but is directly related to the pressure fluctuations and is, as a result, connected with any acoustic-type motion that may occur. We therefore refer to the latter disturbance as 'acoustic' though we realize that it will occur even when the fluid is incompressible. Finally, the fluctuations in entropy are decoupled from the velocity and pressure fluctuations, but do produce density fluctuations, and are also 'frozen in the flow'. It is important to notice that each of these three modes of motion is itself a solution to the governing equations and can therefore be imposed on the flow independently of the others.

Since we are considering the upstream region of the flow, the 'acoustic' disturbances will always correspond to actual acoustic waves. We are at liberty to impose arbitrarily the portion of the acoustic waves propagating inwards, towards the obstacle but the

† A list of some of the more commonly used symbols is given in appendix E.

outward-propagating waves must be determined by the solution. We are not interested in imposing an incident acoustic field on the flow but we shall otherwise consider the most general type of incident disturbance field, which, as we have seen, consists of the gust and entropy modes.

Since these disturbances are both frozen in the flow (i.e. they appear steady to an observer moving with the mean flow) and the acoustic field will decay at infinity in the absence of incident acoustic waves, the upstream velocity field must be of the form

$$\mathbf{v}(x,y,z,t) = \hat{\mathbf{i}}U_\infty + \mathbf{u}_\infty(x-U_\infty t, y, z), \quad \nabla \cdot \mathbf{u}_\infty = 0 \quad \text{as} \quad x \to -\infty, \tag{2.1}$$

where t is the time, (x, y, z) are Cartesian co-ordinates and $\hat{\mathbf{i}}$ is a unit vector in the x direction, which we have assumed to coincide with that of the mean flow. The entropy S must be of the form

$$S = s_\infty(x-U_\infty t, y, z) \quad \text{as} \quad x \to -\infty. \tag{2.2}$$

Finally, since neither the entropy nor the vortical mode will produce any pressure fluctuations we must require that

$$p \to p_\infty = \text{constant} \quad \text{as} \quad x \to -\infty. \tag{2.3}$$

The functions $\mathbf{u}_\infty$ and s_∞ are boundary conditions that can be imposed on the flow. They are often taken to be stationary random functions of their arguments in order to represent turbulence in the incident stream. Liepmann (1952) was probably the first to use statistical methods in conjunction with a flow of the type of (2.1) to represent an incident homogeneous turbulence field. A more detailed treatment has recently been given by Hunt (1973).

2.1. *Derivation of linearized equations*

We shall, for simplicity, restrict our attention to an ideal gas, so that the pressure p, density ρ and temperature T are related through a gas constant R by $p/\rho = RT$. We also suppose that the specific heats are constant, so that the change in the entropy S between any two states 1 and 2 is related to the corresponding pressures and densities by

$$S_1 - S_2 = c_v \ln(p_1/p_2) - c_p \ln(\rho_1/\rho_2). \tag{2.4}$$

Since the flow is assumed to be inviscid and non-heat-conducting the governing momentum, continuity and energy equations can be written as

$$\rho D\mathbf{v}/Dt = -\nabla p, \tag{2.5}$$

$$D\rho/Dt + \rho\nabla \cdot \mathbf{v} = 0, \tag{2.6}$$

$$DS/Dt = 0, \tag{2.7}$$

respectively, where $\mathbf{v}$ is the velocity and

$$D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$$

is the substantive derivative.

We shall suppose that any shock waves that may exist will always remain quite weak. Then if $\mathbf{u}_\infty$ and s_∞ were zero the imposed velocity $\hat{\mathbf{i}}U_\infty$ would produce a steady flow field $\mathbf{U} = \{U_x, U_y, U_z\}$ that could be expressed in terms of a potential $\Phi(x,y,z)$ by

$$\mathbf{U} = \nabla\Phi. \tag{2.8}$$

The entropy would be everywhere equal to a constant, which we can take without loss of generality to be zero. Then (2.4) shows that the pressure p_0 and density ρ_0 would be related by

$$p_0/\rho_0^{\kappa} \equiv \text{constant}, \tag{2.9}$$

where $\kappa = c_p/c_v$ is the ratio of the specific heats.

In the general case where $\mathbf{u}_\infty$ and s_∞ are small (relative to U_∞ and c_p respectively) but non-zero, (2.5)–(2.7) will possess a solution of the form

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{U}(\mathbf{x}) + \mathbf{u}(\mathbf{x}, t), \quad p = p_0(\mathbf{x}) + p'(\mathbf{x}, t), \\ \rho &= p_0(\mathbf{x}) + \rho'(\mathbf{x}, t), \quad S = s'(\mathbf{x}, t), \end{aligned}\right\} \tag{2.10}$$

where we have put

$$\mathbf{x} = (x_1, x_2, x_3), \quad x_1 = x, \quad x_2 = y, \quad x_3 = z$$

and both $\mathbf{u}$ and the primed quantities denote small perturbations of the order of $\mathbf{u}_\infty$ and s_∞.

If the obstacle is thin in one transverse dimension (i.e. if it has small fineness ratio) we shall suppose that either (i) the upstream mean-flow Mach number M_∞ is always sufficiently far from unity to ensure that (Landahl 1961, pp. 3–8)

$$|M_\infty - 1| \text{ not} \ll 1 \tag{2.11a}$$

or (ii) that the characteristic frequency ω_c of the unsteady motion is large enough to ensure that

$$\omega_c l/U_\infty \gg |1 - M_\infty|, \tag{2.11b}$$

where l is a characteristic dimension of the obstacle in the upstream mean-flow direction. Then (2.5)–(2.7) can be linearized about the mean flow (2.8) for obstacles of any fineness ratio. We can therefore neglect squares of small quantities and subtract out the mean-flow equations

$$\rho_0 \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p_0, \quad \mathbf{U} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{U} = 0 \tag{2.12a, b}$$

to obtain

$$\rho_0(D_0\mathbf{u}/Dt + \mathbf{u} \cdot \nabla \mathbf{U}) + \rho' \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p', \tag{2.13}$$

$$D_0 \rho'/Dt + \rho' \nabla \cdot \mathbf{U} + \nabla \cdot (\rho_0 \mathbf{u}) = 0, \tag{2.14}$$

$$D_0 s'/Dt = 0, \tag{2.15}$$

where $D_0/Dt = \partial/\partial t + \mathbf{U} \cdot \nabla$ denotes the convective derivative associated with the basic steady flow.

The derivation of the final results can be simplified by first transforming (2.13) into a more convenient form. In order to do this we apply (2.4) between the state (denoted by the subscript 0) of the mean background flow at the point $\mathbf{x}$ and the actual state at that point and then neglect the squares of small quantities to show that

$$s' = -c_p \rho'/\rho_0 + c_v p'/p_0. \tag{2.16}$$

Then it follows from (2.12a) that (2.13) can be written as

$$\frac{D_0 \mathbf{u}}{Dt} + \mathbf{u} \cdot \nabla \mathbf{U} - \frac{s'}{c_p} \mathbf{U} \cdot \nabla \mathbf{U} = \frac{p'}{\kappa \rho_0 p_0} \nabla p_0 - \frac{1}{\rho_0} \nabla p'.$$

Equations (2.9) and (2.15) therefore imply that

$$D_0 \mathbf{u}^*/Dt + \mathbf{u}^* \cdot \nabla \mathbf{U} = -\nabla(p'/\rho_0), \tag{2.17}$$

where

$$\mathbf{u}^* \equiv \mathbf{u} - (s'/2c_p)\mathbf{U}. \tag{2.18}$$

This equation can be used in place of the linearized momentum equation (2.13). It has the same form as the momentum equation for a barotropic flow but with the actual velocity replaced by what we might call the effective velocity $\mathbf{u}^*$.

It is also helpful to transform the continuity equation (2.14). To this end we note that (2.12*b*) can be combined with (2.14) to obtain

$$\frac{D_0}{Dt}\frac{\rho'}{\rho_0}+\frac{1}{\rho_0}\nabla\cdot\rho_0\mathbf{u} = 0.$$

Then since the speed of sound c_0 of the mean background flow is equal to $(c_p\,p_0/c_v\,\rho_0)^{\frac{1}{2}}$, (2.16) can be written as

$$\frac{\rho'}{\rho_0} = \frac{p'}{c_0^2\rho_0}-\frac{1}{c_p}s'.$$

It therefore follows from (2.15) that

$$\frac{D_0}{Dt}\left(\frac{p'}{c_0^2\rho_0}\right)+\frac{1}{\rho_0}\nabla\cdot\rho_0\mathbf{u} = 0. \tag{2.19}$$

2.2. *Integration of linearized equations*

We have now replaced the original system (2.13)–(2.15) by (2.15), (2.17) and (2.19). Equation (2.15) is a first-order linear partial differential equation that can be easily integrated. In order to accomplish this we notice that, as pointed out by Darwin (1953), the equations

$$\frac{dx}{U_x} = \frac{dy}{U_y} = \frac{dz}{U_z} = dt \tag{2.20}$$

for the streamlines of the basic flow $\mathbf{U} = (U_x, U_y, U_z)$ (which are the characteristic equations of this first-order partial differential equation) possess two functionally independent integrals $Y(x,y,z)$ and $Z(x,y,z)$ such that

$$Y\to y,\quad Z\to z\quad \text{as}\quad x\to-\infty. \tag{2.21}$$

Moreover, the equations of the streamlines $y = y_s(x, Y, Z)$ and $z = z_s(x, Y, Z)$ [which are the solutions of (2.20)] can be obtained by solving the equations $Y(x,y,z) =$ constant and $Z(x,y,z) =$ constant for y and z as a function of x (which, in geometric terms, means that the mean-flow streamlines lie along the intersections of surfaces $Y =$ constant and $X =$ constant). For two-dimensional potential flow there exists a stream function Ψ and we can take Y and Z to be Ψ/U_∞ and z, respectively.

Finally we introduce Lighthill's (1956) 'drift' function

$$\Delta(x,y,z) = \frac{x}{U_\infty}+\int_{-\infty}^{x}\left[\frac{1}{U_x(x', y_s(x', Y, Z), z_s(x', Y, Z))}-\frac{1}{U_\infty}\right]dx', \tag{2.22}$$

whose difference between any two points on a streamline is equal to the time it takes a fluid particle to traverse the distance between those points. The y and z dependence of this equation results from the y and z dependence of $Y(x,y,z)$ and $Z(x,y,z)$. Then $t-\Delta(x,y,z)$ is the third independent integral of (2.20) and it follows from elementary differential equation theory that the vector

$$\mathbf{X} = \{X_1, X_2, X_3\}, \tag{2.23a}$$

where

$$X_1 = U_\infty\Delta,\quad X_2 = Y,\quad X_3 = Z, \tag{2.23b}$$

must satisfy†

$$\frac{D_0}{Dt}(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t) = 0 \tag{2.24}$$

and that the most general solution to (2.15) can be written as

$$s' = \hat{S}(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t), \tag{2.25}$$

where $\hat{S}$ is an arbitrary function of its arguments.

We shall now integrate (2.17). Let $U_1 = U_x$, $U_2 = U_y$ and $U_3 = U_z$. Then since (2.8) implies that $\partial U_i/\partial x_j - \partial U_j/\partial x_i = 0$ (for $i = 1, 2, 3$) it is easy to see that we can satisfy (2.17) for any function ϕ of $\mathbf{x}$ and t by putting

$$p' = -\rho_0 D_0\phi/Dt \tag{2.26}$$

and $\mathbf{u}^* = \nabla\phi$. However, this is certainly not the most general solution to (2.17). In fact, it cannot even be made to satisfy the upstream boundary conditions (2.1) and (2.3). In order to obtain the general solution we note that

$$\partial U_i/\partial x_j - \partial U_j/\partial x_i = 0,$$

and use (2.24) to show that the homogeneous equation

$$D_0\mathbf{u}^{(H)}/Dt + \mathbf{u}^{(H)}.\nabla\mathbf{U} = 0, \tag{2.27}$$

where $\mathbf{u}^{(H)} = (u_1^{(H)}, u_2^{(H)}, u_3^{(H)})$, possesses the solution

$$u_i^{(H)} = \mathscr{A}(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t).\partial\mathbf{X}/\partial x_i \quad \text{for} \quad i = 1, 2, 3,$$

where $\mathscr{A}$ is an arbitrary vector function of its argument $\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t$. Since this solution involves three arbitrary scalar functions, it is also the most general solution of (2.27). The vector $\mathbf{u}^{(H)}$ is therefore a homogeneous solution of (2.17) and the most general solution to this equation is given by (2.26) and $\mathbf{u}^* = \nabla\phi + \mathbf{u}^{(H)}$. Hence it follows from (2.18) that the most general solution to the original momentum equation (2.13) is given by (2.26) and

$$\mathbf{u} = \nabla\phi + \mathbf{u}^{(I)}, \tag{2.28}$$

where $\mathbf{u}^{(I)} = (u_1^{(I)}, u_2^{(I)}, u_3^{(I)})$ is defined by

$$u_i^{(I)} = (s'/2c_p)\,U_i + \mathscr{A}(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t).\partial\mathbf{X}/\partial x_i \quad \text{for} \quad i = 1, 2, 3. \tag{2.29}$$

Equations (2.13) and (2.15) are identically satisfied by (2.25), (2.26) and (2.28) for any function ϕ of $\mathbf{x}$ and t and any functions $\hat{S}$ and $\mathscr{A}$ of $\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t$. However, we must still satisfy the continuity equation (2.14) or equivalently (2.19). To this end we substitute (2.26) and (2.28) into the latter equation to obtain

$$\frac{D_0}{Dt}\left(\frac{1}{c_0^2}\frac{D_0\phi}{Dt}\right) - \frac{1}{\rho_0}\nabla.(\rho_0\nabla\phi) = \frac{1}{\rho_0}\nabla.\rho_0\mathbf{u}^{(I)}. \tag{2.30}$$

Then since ϕ is arbitrary we can ensure that the continuity equation will be satisfied by requiring that ϕ satisfies the linear inhomogeneous equation (2.30) with source term $\rho_0^{-1}\nabla.\rho_0\mathbf{u}^{(I)}$. The original system of equations (2.13)–(2.15) will then be satisfied by (2.25), (2.26) and (2.28) for any choice of the functions $\hat{S}$ and $\mathscr{A}$. Since (2.30) is a

† Equation (2.24) merely states that each component of the vector $\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t$ remains constant for an observer moving with the mean flow. The components X_2 and X_3, which remain constant along the mean-flow streamlines, will certainly have this property and the remarks following (2.22) show that $X_1 - U_\infty t$ also has this property.

linear inhomogeneous wave equation, we can require that ϕ satisfies an appropriate linear boundary condition (to be discussed below) on the surface of the obstacle and, as long as $\nabla . \rho_0 \mathbf{u}^{(I)}$ goes to zero as $x_1 \to -\infty$, we can also require that

$$\phi(\mathbf{x}, t) \to 0 \quad \text{as} \quad x_1 \to -\infty. \tag{2.31}$$

We shall now show that the functions $\hat{S}$ and $\mathscr{A}$ can be adjusted to make the above solution satisfy the upstream boundary conditions (2.1)–(2.3). To this end notice that, since (2.21), (2.22), and (2.23) imply that $\mathbf{X} - \hat{\mathbf{i}} U_\infty t \to \mathbf{x} - \hat{\mathbf{i}} U_\infty t$ as $x_1 \to -\infty$, it follows from (2.29) and (2.25) that $\mathbf{u}^{(I)}$ will approach $\mathbf{u}_\infty(\mathbf{x} - \hat{\mathbf{i}} U_\infty t)$ as $x_1 \to -\infty$ if we put

$$\mathscr{A}(\mathbf{X} - \hat{\mathbf{i}} U_\infty t) = \mathbf{u}_\infty(\mathbf{X} - \hat{\mathbf{i}} U_\infty t) - \hat{\mathbf{i}}(U_\infty / 2c_p)\, \hat{S}(\mathbf{X} - \hat{\mathbf{i}} U_\infty t).$$

But since $\rho_0 \to$ constant as $x_1 \to -\infty$ and $\nabla . \mathbf{u}_\infty = 0$, the right side of (2.30) will vanish as $x_1 \to -\infty$ and we can therefore require that ϕ satisfies the boundary condition (2.31). Then since $\mathbf{U} \to \hat{\mathbf{i}} U_\infty$ and $p_0 \to$ constant as $x_1 \to -\infty$, it follows from (2.10), (2.26), (2.28) and (2.29) that the upstream boundary conditions (2.1) and (2.3) are satisfied. Finally, (2.25) implies that we can satisfy the upstream condition (2.2) by putting

$$\hat{S}(\mathbf{X} - \hat{\mathbf{i}} U_\infty t) = s_\infty(\mathbf{X} - \hat{\mathbf{i}} U_\infty t).$$

We have now shown that the solution to the linearized equations (2.13)–(2.15) that satisfies the upstream boundary conditions (2.1)–(2.3) and still retains enough generality to satisfy an appropriate linear boundary condition on the surface of the obstacle (to be discussed subsequently) is given by (2.28), (2.26) and

$$s' = s_\infty(\mathbf{X} - \hat{\mathbf{i}} U_\infty t), \tag{2.32}$$

where $$u_i^{(I)} \equiv \mathbf{u}_\infty(\mathbf{X} - \hat{\mathbf{i}} U_\infty t) . \frac{\partial \mathbf{X}}{\partial x_i} + \frac{1}{2c_p} s_\infty(\mathbf{X} - \hat{\mathbf{i}} U_\infty t) \left(U_i - U_\infty^2 \frac{\partial \Delta}{\partial x_i} \right), \quad i = 1, 2, 3, \tag{2.33}$$

and ϕ is a solution to the inhomogeneous wave equation (2.30) that satisfies (2.31) at infinity and an appropriate linear boundary condition on the surface of the obstacle. When the boundaries are rigid this condition is the result of requiring that the normal component of the velocity should vanish at the surface, in which case it follows from (2.28) that the boundary condition can be written as

$$\hat{\mathbf{n}} . \nabla \phi \to -\hat{\mathbf{n}} . \mathbf{u}^{(I)} \quad \text{(as } \mathbf{x} \text{ approaches the solid surface)}, \tag{2.34}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface.

These results show that the velocity field is the sum of (i) a 'known' velocity $\mathbf{u}^{(I)}$ that can be calculated by quadratures from the basic potential flow velocity field $\mathbf{U}$ and the imposed upstream velocity and entropy fields $\mathbf{u}_\infty$ and s_∞ and (ii) a velocity $\nabla \phi$ that is the gradient of a perturbation potential, which determines the pressure fluctuations in the usual way via (2.26). This potential is, in turn, determined by the inhomogeneous wave equation (2.30), the left side of which is simply the wave equation for a slightly unsteady potential flow while the strength of the *dipole* source term on the right side is effectively the known incident velocity field $\mathbf{u}^{(I)}$.

2.3. *Interpretation of results*

It is clear from (2.28) that the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is completely determined by $\mathbf{u}^{(I)}$ (i.e. $\boldsymbol{\omega} = \nabla \times \mathbf{u}^{(I)}$), the first term of (2.33) contributing a term that represents the effect of the imposed upstream vorticity field

$$\boldsymbol{\omega}_\infty(\mathbf{x} - \hat{\mathbf{i}} U_\infty t) \equiv \nabla \times \mathbf{u}_\infty(\mathbf{x} - \hat{\mathbf{i}} U_\infty t)$$

and the second term of (2.33) contributing a term that represents the vorticity generated by the incident entropy field. In fact, it is shown in appendix D that the contribution from the first term in (2.33) is simply

$$\omega_i^{(1)} = \left|\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right| \boldsymbol{\omega}_\infty(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t)\,.\frac{\partial \mathbf{x}}{\partial a_i}, \quad i = 1,2,3, \tag{2.35}$$

where the $a_i \equiv X_i - \delta_{i,1} U_\infty t$ are essentially Lagrangian co-ordinates and $|\partial \mathbf{a}/\partial \mathbf{x}|$ denotes the determinant of the $\partial a_i/\partial x_j$. This result is easily recognized as the linearized version of Cauchy's equation for the vorticity field in a barotropic flow. However, we have now shown that it represents the portion of the vorticity arising from the imposed upstream vortical velocity field (i.e. the portion of the vorticity field that is directly imposed on the flow) even when the flow is non-barotropic. (A corresponding result holds even when the flow is not linearized.)

In flows that exist for all time we can, with suitable caution, always represent the incident disturbance field $\mathbf{u}_\infty(\mathbf{x}-\mathbf{\hat{\imath}}\mathbf{U}_\infty t)$, $s_\infty(\mathbf{x}-\mathbf{\hat{\imath}}U_\infty t)$ by the generalized Fourier integrals

$$\mathbf{u}_\infty(\mathbf{x}-\mathbf{\hat{\imath}}U_\infty t) = \int \mathbf{A}(\mathbf{k}) \exp(i\mathbf{k}\,.\,\mathbf{x}')\,d\mathbf{k}, \quad s_\infty(\mathbf{x}-\mathbf{\hat{\imath}}U_\infty t) = \int B(\mathbf{k}) \exp(i\mathbf{k}\,.\,\mathbf{x}')\,d\mathbf{k},$$

where $\mathbf{x}' \equiv \mathbf{x} - \mathbf{\hat{\imath}}U_\infty t$ and in order to ensure that $\mathbf{u}_\infty$ is solenoidal we must require that $\mathbf{A}\,.\,\mathbf{k} = 0$ for all $\mathbf{k}$. Then since (2.33), (2.30) and (2.34) are linear in ϕ, $\mathbf{u}_\infty$ and s_∞ we can find the solution ϕ for any upstream disturbance field $\mathbf{u}_\infty$, s_∞ simply by superposing solutions to the corresponding problem for an arbitrary incident harmonic disturbance field

$$\mathbf{u}_\infty = \mathbf{A} \exp\{i(\mathbf{k}\,.\,\mathbf{x} - k_1 U_\infty t)\}, \quad s_\infty = B \exp\{i(\mathbf{k}\,.\,\mathbf{x} - k_1 U_\infty t)\}. \tag{2.36}$$

Hence, in many cases, we need consider incident disturbances of this type only.

The portion $\mathbf{u}^{(I)}$ of (2.28) represents the direct effect of the upstream disturbances $\mathbf{u}_\infty$ and s_∞. Suppose first that the flow is barotropic, i.e. $s_\infty = 0$. Then $\mathbf{u}^{(I)}$ is equal to $\mathbf{u}^{(H)}$ [with the arbitrary vector $\mathscr{A}$ set equal to $\mathbf{u}_\infty$; see (2.29), (2.33) and the equation following (2.27)] and therefore satisfies (2.27). If the second term in this equation were zero (so that $D_0 \mathbf{u}^{(I)}/Dt \equiv 0$), $\mathbf{u}^{(I)}$ would be equal to $\mathbf{u}_\infty(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t)$ [see (2.24)] and would therefore remain constant for an observer who moved along the mean streamlines at the mean flow velocity [since Y and Z are constant along these streamlines while the change in Δ between any two points on a streamline is equal to the time it takes a fluid particle to traverse the distance between them; see (2.23*b*)]. But since the mean velocity is different on different streamlines and since the streamlines themselves converge and diverge the spatial distribution of $\mathbf{u}^{(I)}$ will be different near the obstacle from its form at upstream infinity. In this way the mean flow is able to distort (i.e. to alter the character of) the incident disturbance field $\mathbf{u}_\infty$ even when the second term in (2.27) is absent.

The nature of the alteration is best appreciated by considering the simple harmonic velocity disturbance in (2.36). Then $\mathbf{u}^{(I)}$ (with the $\partial \mathbf{X}/\partial x_i$ factor omitted) will equal $\mathbf{A} \exp\{i\mathbf{k}\,.\,(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t)\}$, which shows that the amplitude of this disturbance is unaltered while its phase is changed from $\mathbf{k}\,.\,(\mathbf{x}-\mathbf{\hat{\imath}}U_\infty t)$ to $\mathbf{k}\,.\,(\mathbf{X}-\mathbf{\hat{\imath}}U_\infty t)$. Moreover its propagation direction (i.e. the direction normal to its phase surface) is $k_i \nabla X_i$ while its phase velocity is $k_1 U_\infty/|k_i \nabla X_i|$.

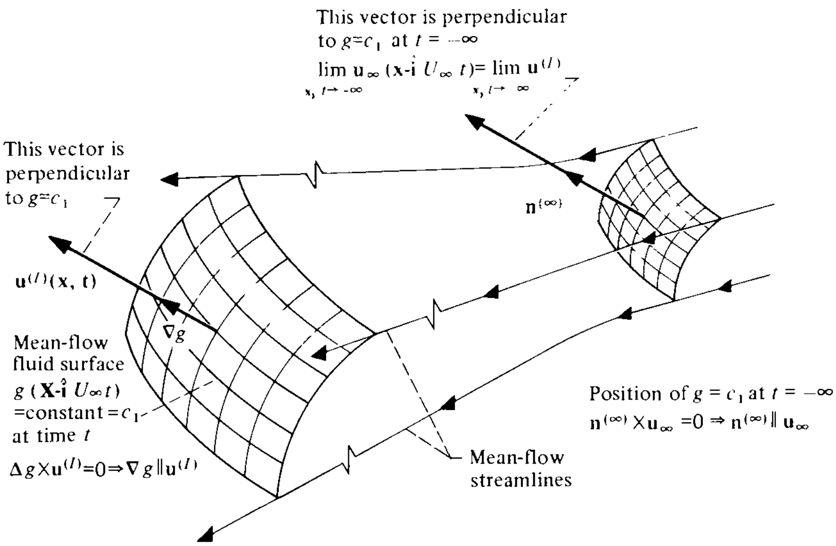

FIGURE 1. Evolution of $\mathbf{u}^{(I)}$ for barotropic flow (∇g perpendicular to g = constant for any function g).

The net effect of the second term in (2.27) is to change $\mathbf{u}^{(I)}$ from $\mathbf{u}_\infty(\mathbf{X}-\hat{\mathbf{i}}U_\infty t)$ into the solution

$$\mathbf{u}_i^{(I)} = \mathbf{u}_\infty(\mathbf{X}-\hat{\mathbf{i}}U_\infty t)\,.\,\partial\mathbf{X}/\partial x_i, \quad i = 1, 2, 3,$$

of the complete equation. This formula implies that *the vector* $\mathbf{u}^{(I)}$ *always remains perpendicular to the same fluid surface*† *as it is convected downstream by the mean flow.*

In order to show this notice (i) that, since $\mathbf{X}-\hat{\mathbf{i}}U_\infty t$ remains constant when following a fluid particle, any mean flow fluid surface can be represented by an equation of the form $g(\mathbf{X}-\hat{\mathbf{i}}U_\infty t)$ = constant and (ii) that the chain rule for partial differentiation implies that $\nabla g = n_i^{(\infty)}(\mathbf{X}-\hat{\mathbf{i}}U_\infty t)\,\nabla X_i$, where $n_i^{(\infty)}(\mathbf{X}-\hat{\mathbf{i}}U_\infty t) \equiv \partial g(\mathbf{X}-\hat{\mathbf{i}}U_\infty t)/\partial X_i$ (see figure 1). Hence it follows‡ that $\nabla g \times \mathbf{u}^{(I)}$ will equal zero if

$$\mathbf{u}_\infty(\mathbf{X}-\hat{\mathbf{i}}U_\infty t)\times\mathbf{n}^{(\infty)}(\mathbf{X}-\hat{\mathbf{i}}U_\infty t) = 0.$$

But since $\mathbf{X}-\hat{\mathbf{i}}U_\infty t$ remains constant when following a fluid particle and reduces to $\mathbf{x}-\hat{\mathbf{i}}U_\infty t$ when that particle is far upstream, $\mathbf{n}^{(\infty)}$ will equal ∇g when the surface g = constant is far upstream and will, therefore, remain perpendicular to the upstream configuration of this surface. Then since vectors with a zero cross-product are parallel, $\mathbf{u}^{(I)}$ will be perpendicular to g = constant at $(\mathbf{x}, t)$ if $\mathbf{u}_\infty$ was perpendicular to this surface when it was far upstream.

Now we can always think of the vector $\mathbf{u}^{(I)}(\mathbf{x}, t)$ as being permanently attached to any mean-flow fluid surface, say $g(\mathbf{X}-\hat{\mathbf{i}}U_\infty t)$ = constant, that passes through the point $\mathbf{x}$ at the time t. Then since $\mathbf{u}^{(I)}\rightarrow\mathbf{u}_\infty$ at upstream infinity, the preceding result shows that $\mathbf{u}^{(I)}$ will be perpendicular to the surface g = constant if it was perpendicular to it at the time $t = -\infty$, when it was at upstream infinity. This proves the assertion.

If it were not for the factor $\partial\mathbf{X}/\partial x_i$ and consequently the second term in (2.27), the orientation of $\mathbf{u}^{(I)}$ would remain unchanged as it moved downstream. The second term

† A fluid surface is one that always consists of the same fluid particles.

‡ By inserting the expressions for ∇g and $\mathbf{u}^{(I)}$ into the cross-product and interchanging dummy indices in the result.

in (2.27) is therefore responsible for keeping $\mathbf{u}^{(I)}$ perpendicular to the mean-flow fluid surfaces.

This term also causes $\mathbf{u}^{(I)}$ to change in magnitude. But since (2.24) implies that $\mathbf{U}.\nabla X_i = D_0 X_i/Dt = U_\infty \delta_{1i}$,

$$u_s^{(I)} \equiv \frac{\mathbf{U}}{|\mathbf{U}|}.\mathbf{u}^{(I)} = \frac{U_\infty}{|\mathbf{U}|} u_{\infty 1}(\mathbf{X} - \hat{\mathbf{\imath}} U_\infty t),$$

which shows that $|\mathbf{U}|$ times the streamwise component of $\mathbf{u}^{(I)}$ (i.e. the component along the mean-flow streamlines) now remains unchanged relative to an observer moving with the flow. This component of $\mathbf{u}^{(I)}$ is therefore (i) unaffected by the transverse distortion components $u_{\infty 2}$ and $u_{\infty 3}$ and (ii) becomes infinite at all stagnation points of the mean flow. The cross-stream components of $\mathbf{u}^{(I)}$, on the other hand, are affected by all components of $\mathbf{u}_\infty$.

The streamwise velocity fluctuation $|\mathbf{U}|u_s^{(I)}$ is not the only quantity that remains 'frozen in the flow'. In order to show this note that the results of appendix D (i.e. equation (D 2) together with the Laplace expansion formula for determinants) imply that

$$\mathbf{u}^{(I)}.\boldsymbol{\omega} = \mathbf{u}^{(I)}.(\nabla \times \mathbf{u}^{(I)}) = \mathbf{u}^{(I)}.\boldsymbol{\omega}^{(1)} = |\partial \mathbf{X}/\partial \mathbf{x}|\, \mathbf{u}_\infty(\mathbf{X} - \hat{\mathbf{\imath}} U_\infty t).\boldsymbol{\omega}_\infty(\mathbf{X} - \hat{\mathbf{\imath}} U_\infty t).$$

Hence, using the well-known result $|\partial \mathbf{X}/\partial \mathbf{x}| = \rho_0/\rho_\infty$, we find that

$$\mathbf{u}^{(I)}.\boldsymbol{\omega}/\rho_0 = \mathbf{u}_\infty(\mathbf{X} - \hat{\mathbf{\imath}} U_\infty t).\boldsymbol{\omega}_\infty(\mathbf{X} - \hat{\mathbf{\imath}} U_\infty t)/\rho_\infty$$

also remains constant for an observer moving with the mean flow. Then when the upstream velocity distortion is initially complex lamellar (i.e. when† $\mathbf{u}^{(I)}.\boldsymbol{\omega} = 0$ at upstream infinity) $\mathbf{u}^{(I)}$ will satisfy that condition at all points of the flow. However, it should not be concluded from this and the fact that $\nabla\phi$ is itself lamellar that the complete velocity field $\nabla\phi + \mathbf{u}^{(I)}$ will then be complex lamellar, since $\nabla\phi.\boldsymbol{\omega}$ will not in general be zero.

It is again instructive to consider the simple harmonic disturbance (2.36). Then the effect of the factor $\partial\mathbf{X}/\partial x_i$, and consequently the second term in (2.27), is to change the amplitude of the disturbance from its initial value of $\mathbf{A}$ to $A_i \nabla X_i$ while leaving its phase the same as in the previous case.

Thus the phase changes are produced by the first term in (2.27) while the amplitude changes are produced by the second. The amplitude varies from $\mathbf{A}$ to $A_i \nabla X_i$ while the propagation direction (i.e. the phase-surface normal) varies from $\mathbf{k}$ to $k_i \nabla X_i$. The streamwise amplitude component $(\mathbf{U}/|\mathbf{U}|).[(\nabla X_i) A_i]$ is equal to $U_\infty A_1/|\mathbf{U}|$ and therefore varies in inverse proportion to $|\mathbf{U}|$ and is independent of the transverse amplitude components A_2 and A_3.

Since $\mathbf{u}_\infty(\mathbf{x} - \hat{\mathbf{\imath}} U_\infty t)$ has zero divergence, $\mathbf{k}.\mathbf{A} = 0$ and the initial propagation direction of the upstream disturbance is perpendicular to its amplitude, i.e. the upstream disturbance is a transverse wave. But since $k_i \nabla X_i$ will not in general be perpendicular to $A_i \nabla X_i$, this condition will not be maintained near the obstacle. It can also be seen that $\mathbf{u}^{(I)}$ *will not in general remain divergence free* in this region.

Since $\nabla \times \mathbf{u}_\infty$ is now equal to $i\mathbf{k} \times \mathbf{A} \exp\{i\mathbf{k}.(\mathbf{x} - \hat{\mathbf{\imath}} U_\infty t)\}$ and $(\mathbf{k} \times \mathbf{A}).\mathbf{A} = 0$, the

† A vector field $\mathbf{V}$ is said to be *complex lamellar* if it is everywhere perpendicular to a one-parameter family of surfaces (i.e. if there exist real scalar functions f and g such that $\mathbf{V} = g\nabla f$). However, it can be shown (Truesdell 1954, p. 23) that this is entirely equivalent to the requirement that $\mathbf{V}.(\nabla \times \mathbf{V}) = 0$. Complex-lamellar fields have been extensively studied in the fluid-mechanics context (see Truesdell 1954, p. 38 for references) and their properties are well understood.

upstream distortion field is complex lamellar in this case and $\mathbf{u}^{(I)}$ is therefore complex lamellar everywhere in the flow.

Now suppose that $s_\infty \neq 0$. Then (2.33) implies that the incident velocity field $\mathbf{u}^{(I)}$ will contain the additional term

$$\mathbf{u}^{(s)} \equiv \frac{s_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t)}{2c_p}(\mathbf{U} - U_\infty^2 \nabla\Delta),$$

which, in view of (2.8), can also be written as

$$\mathbf{u}^{(s)} = \frac{s_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t)}{2c_p}\nabla(\Phi - U_\infty^2 \Delta).$$

This shows that the entropic portion of $\mathbf{u}^{(I)}$ is everywhere perpendicular to the surfaces of constant $\Phi - U_\infty^2 \Delta$ and is therefore complex lamellar.

It follows from (2.24) and (2.23*b*) that the streamwise component $(\mathbf{U}/|\mathbf{U}|).\mathbf{u}^{(s)}$ of $\mathbf{u}^{(s)}$ is equal to

$$\frac{s_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t)}{2c_p|\mathbf{U}|}(|\mathbf{U}|^2 - U_\infty^2).$$

But the mean-flow energy equation $|\mathbf{U}|^2 - U_\infty^2 = 2c_p(T_\infty - T)$, where T is the mean temperature and T_∞ is its value at upstream infinity, implies that this can also be written as

$$\frac{s_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t)}{|\mathbf{U}|}(T_\infty - T).$$

These results show that the streamwise component of $\mathbf{u}^{(s)}$ will become infinite at the mean-flow stagnation points and will become large at any sharp turns in the flow since the latter cause $|\mathbf{U}|$ to become large.

Since p' goes to zero at upstream infinity, the ideal-gas equation implies that the upstream temperature fluctuation t_∞ is related to the corresponding density fluctuation by $t_\infty/T_\infty = -\lim_{x\to-\infty}(\rho'/\rho_0)$. Hence (2.16) implies that $s_\infty = c_p t_\infty/T_\infty$ and therefore that the streamwise component of $\mathbf{u}^{(s)}$ is related to t_∞ by

$$\mathbf{u}^{(s)}.(\mathbf{U}/|\mathbf{U}|) = \frac{c_p t_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t)}{|\mathbf{U}|}\left(1 - \frac{T}{T_\infty}\right) = \frac{t_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t)}{T_\infty}\frac{1}{2}\frac{(|\mathbf{U}|^2 - U_\infty^2)}{|\mathbf{U}|}.$$

We can get an idea of the relative importance of the entropy fluctuations by comparing this with $u_s^{(I)}$, the streamwise component of $\mathbf{u}^{(I)}$ produced by the upstream velocity fluctuations. The ratio of these two quantities is

$$\frac{c_p}{U_\infty^2}\frac{t_\infty}{T_\infty}\frac{U_\infty}{u_{\infty 1}}(T_\infty - T) = \frac{t_\infty}{T_\infty}\frac{U_\infty}{u_{\infty 1}}\frac{1}{2}\left(\frac{|\mathbf{U}|^2}{U_\infty^2} - 1\right).$$

The streamwise entropy fluctuations will therefore be most important at sharp turns, where the mean velocity $|\mathbf{U}|$ becomes large relative to U_∞. At the other points of the flow, where $|\mathbf{U}|$ is of the same order as U_∞, the entropy fluctuations will be of equal importance to the velocity fluctuations when t_∞/T_∞ is of the order of u_∞/U_∞.

The vorticity $\boldsymbol{\omega}^{(2)}$ associated with $\mathbf{u}^{(s)}$,

$$\boldsymbol{\omega}^{(2)} \equiv \nabla \times \mathbf{u}^{(s)} = (2c_p)^{-1}(\nabla s_\infty) \times \nabla(\Phi - U_\infty^2 \Delta),$$

is perpendicular to $\nabla(\Phi - U_\infty^2 \Delta)$ and therefore also to $\mathbf{u}^{(s)}$. This shows that $\mathbf{u}^{(s)}.\boldsymbol{\omega}^{(2)}$ is everywhere zero for all upstream disturbance fields. However the complete incident distortion $\mathbf{u}^{(I)}$ will not in general have this property even when the initial velocity distortion is complex lamellar.

When (corresponding to the first group of studies alluded to in the introduction) one of the transverse dimensions of the solid obstacle is small, $X_i \simeq x_i$ and $U_i \simeq \delta_{1,i} U_\infty$ for $i = 1, 2, 3$ and $\rho_0 \simeq$ constant. Then $\mathbf{u}^{(I)} \simeq \mathbf{u}_\infty(\mathbf{x} - \hat{\mathbf{i}} U_\infty t)$ and the source term in the wave equation (2.30) will vanish since $\mathbf{u}_\infty$ has zero divergence. Consequently, the portion $\nabla\phi$ of the velocity field associated with the pressure fluctuations through (2.26) becomes decoupled from the incident velocity field $\mathbf{u}^{(I)}$ and we recover the behaviour that was ascribed to the small fluctuations on uniform flows at the beginning of this section.

In the general case, where the obstacle's transverse dimensions cannot be treated as small, the portion $\nabla\phi$ of the velocity field is still associated with the pressure fluctuations via (2.26) but the incident velocity field can now alter ϕ through the source term in the inhomogeneous wave equation (2.30). This occurs because, as we have just seen, the distortion effect of the mean potential flow destroys the initial divergence-free condition exhibited by the incident momentum flux $\rho_0 \mathbf{u}^{(I)}$ at upstream infinity [where it is equal to $\rho_\infty \mathbf{u}_\infty(\mathbf{x} - \hat{\mathbf{i}} U_\infty t)$]. The source term in (2.30) is therefore non-zero near the obstacle, causing ϕ and consequently p' also to be non-zero there. In this way pressure fluctuations are set up to balance the fluctuations in momentum that arise from the distortion of the gust by the potential flow about the obstacle.

On the other hand, $\nabla\phi$ and $\mathbf{u}^{(I)}$ are *not* completely decoupled even when the mean flow is uniform since the right side of the boundary condition (2.34) will not vanish even in this case. However, $\mathbf{u}^{(I)}$ will still be unaffected by the entropy fluctuations, which therefore become decoupled from both the pressure fluctuations $-\rho_0 D_0 \phi / Dt$ and the 'acoustic' portion $\nabla\phi$ of the velocity field.

In the general case, the entropy fluctuations affect the pressure fluctuations and the acoustic portion of the velocity field through both the boundary condition and the source term in (2.30). The solid surface can therefore 'scatter' entropy fluctuations into a propagating sound field.

2.4. *Incompressible limit*

As long as the frequency of the unsteady motion is not too large, the fluid will behave incompressibly when the mean-flow Mach number is sufficiently small. We can then put $\rho_0 =$ constant and $c_0 = \infty$ in (2.30) to obtain

$$\nabla^2\phi = -\nabla \cdot \mathbf{u}^{(I)} \quad \text{as} \quad U_\infty / c_0 \to 0.$$

When there are no upstream entropy fluctuations, this can be combined with (2.33) to obtain

$$\nabla^2\phi = -\frac{\partial}{\partial x_i}\left[\mathbf{u}_\infty(\mathbf{X} - \hat{\mathbf{i}} U_\infty t) \cdot \frac{\partial \mathbf{X}}{\partial x_i}\right] \quad \text{for} \quad s_\infty = 0 \quad \text{and} \quad U_\infty / c_0 \to 0. \tag{2.37}$$

This Poisson equation governs the flows in the second group of studies alluded to be in the introduction. However, it has never actually been used to calculate these flows. It is therefore worth comparing it with the equations used by Hunt (1973). He started from Cauchy's equation (2.35) for the vorticity and represented the velocity field in terms of the vector and scalar potentials Φ_0 and $\mathbf{\Psi}$ with the choice of gauge $\nabla \cdot \mathbf{\Psi} = 0$ to obtain

$$\mathbf{u} = -\nabla\Phi_0 + \nabla \times \mathbf{\Psi}, \quad \boldsymbol{\omega} \equiv \nabla \times \mathbf{u} = -\nabla^2 \mathbf{\Psi}. \tag{2.38}$$

The incompressible continuity equation $\nabla \cdot \mathbf{u} = 0$ then implies $\nabla^2 \Phi_0 = 0$. It therefore follows from (2.35) and (2.38) that Hunt had to solve a set of three Poisson equations and one Laplace equation. The formulation (2.37) has the advantage of reducing the problem to the solution of a single Poisson equation.

2.5. *Extension to two-dimensional lifting surfaces*

There is one important case where the assumption (2.21) (on which the present results are based) is invalid (Goldstein & Atassi 1976). This occurs whenever a two-dimensional obstacle produces lift in a subsonic stream.† Then since the flow is two-dimensional $Y = \Psi/U_\infty$ while it follows from the theory of potential flows that Ψ/U_∞ behaves like

$$y + (\Gamma/2\pi U_\infty) \ln (x^2+y^2)^{\frac{1}{2}} + \text{constant} \quad \text{as} \quad x \to -\infty \tag{2.39}$$

rather than exhibiting the behaviour (2.21). (Here Γ denotes the circulation about the obstacle.) Of course, no real obstacle is two-dimensional and we can always move far enough away so that three-dimensional effects come into play and the behaviour (2.21) is achieved. On the other hand, there are enormous computational simplifications to be gained by assuming that the flow about a body of high aspect ratio can be treated as two-dimensional. Then, since the velocity field (2.28) and (2.33) will still satisfy the linearized momentum equation identically for all choices of u_∞, s_∞ and ϕ and since we can, in principle, always solve the wave equation (2.30) for ϕ subject to the boundary condition (2.34) and thereby ensure that the linearized continuity equation is also satisfied, the formulation given by (2.26), (2.28), (2.30) and (2.32)–(2.34) still provides a solution to the governing equations that has zero normal velocity on the surface of the obstacle. The difficulty is that the boundary conditions (2.1) and (2.2) can no longer be satisfied at $x_1 = -\infty$. This occurs because, as pointed out by Goldstein & Atassi (1976), the basic potential flow decays so slowly that there is no longer a region at infinity that acts like a uniform stream relative to the unsteady motion. But since $\partial Y/\partial x_i$ still approaches $\delta_{2,i}$ while U_i still approaches $\delta_{1,i} U_\infty$ and $\partial \Delta/\partial x_i$ still approaches $\delta_{1,i}/U_\infty$, $\mathbf{u}^{(I)}$ will now behave like

$$\mathbf{u}_\infty(x - U_\infty t, y + (\Gamma/2\pi U_\infty) \ln (x^2+y^2)^{\frac{1}{2}} + \text{constant}, z) \quad \text{as} \quad x \to -\infty.$$

Then since $\rho_0 \to$ constant,

$$\frac{1}{\rho_0} \nabla . \rho_0 \mathbf{u}^{(I)} \to \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial \eta} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right) . \mathbf{u}_\infty(\mathbf{x} - U_\infty t, \eta, z)$$

$$\text{evaluated at} \quad \eta = y + \frac{\Gamma}{2\pi U_\infty} \ln (x^2+y^2)^{\frac{1}{2}} + \text{constant} \quad \text{as} \quad x \to -\infty.$$

But since we have required that $\nabla . \mathbf{u}_\infty(x - U_\infty t, y, z) = 0$, the source term in (2.30) must again vanish as $x \to -\infty$. We can therefore still (at least in the important special case where $\mathbf{u}_\infty = \mathbf{A} \exp\{i(\mathbf{k}.\mathbf{x} - U_\infty k_1 t)\}$ and $s_\infty = B \exp\{i(\mathbf{k}.\mathbf{x} - U_\infty k_1 t)\}$) impose the upstream boundary equation (2.31). Then the pressure fluctuations will vanish and the velocity field will behave like

$$\mathbf{u} \to \mathbf{u}_\infty(x - U_\infty t, y + (\Gamma/2\pi U_\infty) \ln (x^2+y^2)^{\frac{1}{2}} + \text{constant}, z) \quad \text{as} \quad x \to -\infty \tag{2.40}$$

while the entropy fluctuations will behave like

$$s' \to s_\infty(x - U_\infty t, y + (\Gamma/2\pi U_\infty) \ln (x^2+y^2)^{\frac{1}{2}} + \text{constant}, z) \quad \text{as} \quad x \to -\infty. \tag{2.41}$$

† A similar difficulty occurs for flow around a source-like two-dimensional body with significant mean drag (Graham 1976).

Thus, if we want to take advantage of the simplifications that can be achieved from a two-dimensional flow model, we must impose boundary conditions of the type (2.40) and (2.41) in place of the boundary conditions (2.1) and (2.2). If we then wish to suppose that the distortions $\mathbf{u}_\infty$ and s_∞ are imposed at such a great distance upstream that the obstacle appears three-dimensional and the flow behaves like (2.1) and (2.2), we can determine the form of the functions $\mathbf{u}_\infty(x-U_\infty t, y, z)$ and $s_\infty(x-U_\infty, t, y, z)$ from the imposed boundary conditions and then solve the problem corresponding to a two-dimensional potential flow by using these same functions with y replaced by $y+(\Gamma/2\pi U_\infty)\ln(x^2+y^2)^{\frac{1}{2}}+$ an appropriate constant. The actual value of this constant must be found by solving the complete problem or, much more appropriately, by using the method of matched asymptotic expansions with the reciprocal of the aspect ratio of the obstacle taken as a small parameter. However, it turns out that the value of this constant affects only the phase of each of the flow variables and does so by the same constant amount. It will therefore have no influence on any statistical correlations that may be calculated and, since most incident distortion fields are relatively random, will be of little practical interest.

We have shown that the present formulation can be used to simplify problems of the type already solved. But, more important, it can also be used to solve new types of problem which, to our knowledge, cannot be solved by any other method. One such problem is considered in the next section. We suppose that the flow is compressible and that the obstacle causes a non-negligible distortion of the mean flow. Then in order to emphasize compressibility effects, we assume that the motion is supersonic. Other types of flow will be discussed in subsequent papers.

3. Supersonic flow around a cloner

Consider the flow of a uniform supersonic stream along a wall which, as shown in figure 2(*a*), terminates at the point A. The stream will remain uniform until it reaches the Mach line emanating from the point A and then undergo a centred expansion of the Prandtl–Meyer type. Such expansions also occur when a uniform stream flows around a corner as shown in figure 2(*b*) and on the upper surface of wedges and flat plates placed in a uniform supersonic stream as shown in figures 2(*c*) and (*d*). The present analysis will apply to any such flow that involves a Prandtl–Meyer expansion. We suppose that an unsteady velocity field of the type (2.1) is imposed on the uniform flow in the region upstream of the expansion fan and for simplicity require that the entropy fluctuations be zero in this region. Then the unsteady velocity field will be given by (2.1) everywhere upstream of the leading Mach line of the expansion fan. As shown in figure 3, this line emanates from the centre of the expansion and makes an angle with the incident stream equal to the Mach angle. The unsteady flow in the expansion fan downstream of this line can be found by solving (2.30). The Prandtl–Meyer solution for the mean-flow variables $\mathbf{U}$, ρ_0 and c_0 is

$$U_r = (q_0/\gamma)\sin[\gamma(\theta^*-\theta)], \quad U_\theta = -q_0\cos[\gamma(\theta^*-\theta)], \quad U_z = 0, \quad (3.1)\text{–}(3.3)$$

$$c_0 = -U_\theta, \quad (3.4)$$

$$\rho_0/\rho_\infty = [\cos\gamma(\theta^*-\theta)/\cos\gamma(\theta^*-\theta_\infty)]^{2/(\kappa-1)}, \quad (3.5)$$

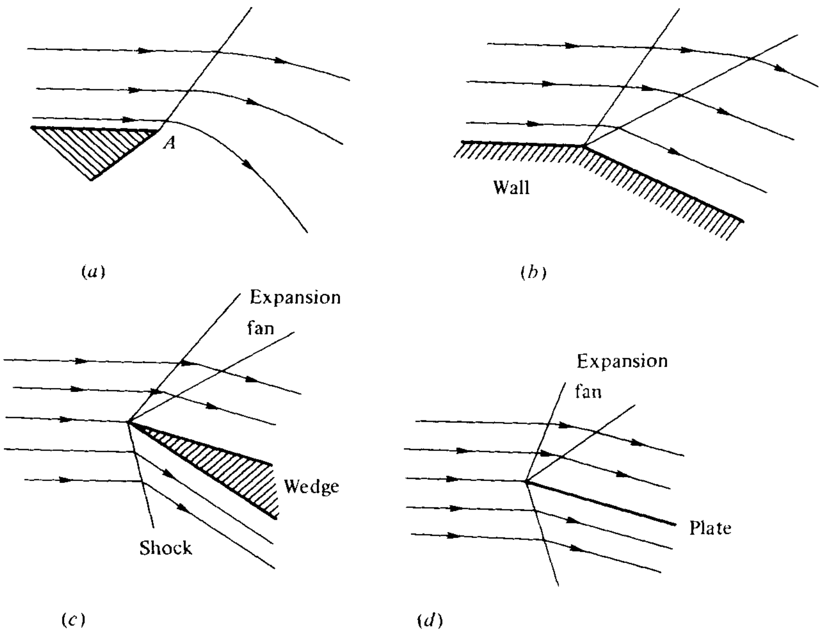

FIGURE 2. Flows involving Prandtl–Meyer expansions. (*a*) Flow past an edge. (*b*) Flow over a wall. (*c*) Flow over a wedge. (*d*) Flow over a flat plate.

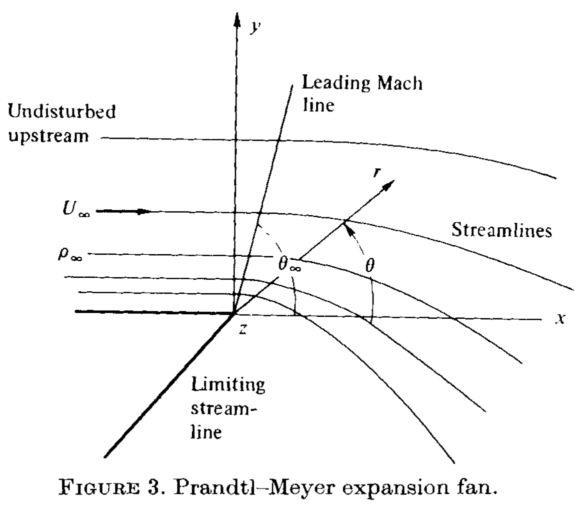

FIGURE 3. Prandtl–Meyer expansion fan.

where U_r, U_θ and U_z are the components of $\mathbf{U}$ in r, θ and z directions in the polar coordinate system shown in figure 3,

$$q_0^2 = \frac{1+\gamma^2(M_\infty^2-1)}{M_\infty^2}\,U_\infty^2, \quad \gamma^2 = \frac{\kappa-1}{\kappa+1}, \qquad (3.6), (3.7)$$

$$\theta^* = \frac{1}{\gamma}\sin^{-1}\left(\frac{\gamma(M_\infty^2-1)^{\frac{1}{2}}}{(\gamma^2(M_\infty^2-1)+1)^{\frac{1}{2}}}\right)+\theta_\infty, \quad \theta_\infty = \sin^{-1}\left(\frac{1}{M_\infty}\right) \qquad (3.8), (3.9)$$

and M_∞ is the Mach number of the incident stream.

In a two-dimensional compressible flow the stream function Ψ is related to the velocity components by

$$U_r = \frac{\rho_\infty}{\rho_0}\frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \quad U_\theta = -\frac{\rho_\infty}{\rho_0}\frac{\partial \Psi}{\partial r}. \tag{3.10}$$

Hence it follows from (3.1), (3.2) and (3.5) that in the present case it must be given by

$$\Psi = q_0 r\{\cos[\gamma(\theta^* - \theta)]\}^{1/\gamma^2}/\cos[\gamma(\theta^* - \theta_\infty)]^{2/(\kappa-1)}. \tag{3.11}$$

It is shown in appendix A that the drift function for the flow in the expansion fan is given by

$$\Delta = \Psi\left\{\frac{\cot\theta_\infty}{U_\infty^2} + \frac{[\cos\gamma(\theta^* - \theta_\infty)]^{2/(\kappa-1)}}{q_0^2}[I(\sin\gamma(\theta^* - \theta)) - I(\sin\gamma(\theta^* - \theta_\infty))]\right\}$$

$$\text{for} \quad \theta > \theta_\infty, \tag{3.12}$$

where

$$I(Z) = \gamma^{-1} Z F(\tfrac{1}{2}, 1 + 1/2\gamma^2; \tfrac{3}{2}; Z^2) \tag{3.13}$$

and F denotes the hypergeometric function in the usual notation.

Rather than considering an arbitrary incident velocity field, we can, as explained by Liepmann (1952), Hunt (1973) and others, carry out the analysis for a single harmonic component

$$\mathbf{u}_\infty(x - U_\infty t, y, z) = \mathbf{A}\exp\{i[k_1(x - U_\infty t) + k_2 y + k_3 z]\} \tag{3.14}$$

of this velocity, where $\mathbf{A}(k_1, k_2, k_3)$ is a constant vector, and, if need be, integrate the final solution over the wavenumbers k_1, k_2 and k_3 to obtain the result corresponding to the required initial disturbance field. The vector $\mathbf{A}$ is not completely arbitrary since in order to ensure that $\mathbf{u}_\infty$ has zero divergence we must take

$$\mathbf{k}.\mathbf{A} = 0, \tag{3.15}$$

where $\mathbf{k} = (k_1, k_2, k_3)$ is the wavenumber vector.

Since the mean flow is two-dimensional, we can take $Y = \Psi/U_\infty$ and $Z = z$. Then since Δ and Ψ are equal to r times a function of θ, it follows from (2.23), (2.33), (3.5), (A 9) and (A 10) that $u_r^{(I)}$, $u_\theta^{(I)}$ and $u_z^{(I)}$, the r, θ and z components of the incident velocity $\mathbf{u}^{(I)}$, are given by

$$u_r^{(I)} = \left[A_1 U_\infty \frac{\Delta}{r} + A_2 \frac{1}{U_\infty}\frac{\Psi}{r}\right]\exp\{i[k_1 U_\infty(\Delta - t) + k_2\Psi/U_\infty + k_3 z]\}, \tag{3.16a}$$

$$u_\theta^{(I)} = \left[A_1 U_\infty \frac{1}{r}\frac{\partial\Delta}{\partial\theta} + A_2\frac{1}{U_\infty}\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right]\exp\{i[k_1 U_\infty(\Delta - t) + k_2\Psi/U_\infty + k_3 z]\}, \tag{3.16b}$$

$$u_z^{(I)} = A_3\exp\{i[k_1 U_\infty(\Delta - t) + k_2\Psi/U_\infty + k_3 z]\} \tag{3.16c}$$

and that the right-side of (2.30) is given by

$$\frac{1}{\rho_0}\nabla.\rho_0\mathbf{u}^{(I)} = \frac{1}{\cos[\gamma(\theta^* - \theta)]}\left[\frac{1}{r}H_0 + iH_1\right]\exp\{i[k_1 U_\infty(\Delta - t) + k_2\Psi/U_\infty + k_3 z]\}, \tag{3.17}$$

where

$$H_0 \equiv \frac{4\sin[\gamma(\theta^* - \theta)]}{\gamma(\kappa + 1)}\left[A_1 U_\infty\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + A_2\frac{1}{U_\infty}\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right] \tag{3.18}$$

and

$$H_1 = \cos[\gamma(\theta^* - \theta)]\left\{A_1 k_1\left[\left(\frac{U_\infty}{r}\frac{\partial\Delta}{\partial\theta}\right)^2 + \left(\frac{U_\infty\Delta}{r}\right)^2\right] + A_2 k_2\left[\left(\frac{1}{U_\infty}\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)^2 + \left(\frac{1}{U_\infty}\frac{\Psi}{r}\right)^2\right]\right.$$

$$\left. + A_3 k_3 + (A_1 k_2 + A_2 k_1)\left[\frac{1}{r^2}\frac{\partial\Delta}{\partial\theta}\frac{\partial\Psi}{\partial\theta} + \frac{\Delta}{r}\frac{\Psi}{r}\right]\right\} \tag{3.19}$$

are functions of θ only.

By introducing polar co-ordinates and using (3.1)–(3.5) we can write the left side of (2.30) as

$$\frac{1}{\cos[\gamma(\theta^*-\theta)]}\left\{\frac{1}{q_0^2}\frac{\partial^2\hat{\phi}}{\partial t^2}-\cos^2[\gamma(\theta^*-\theta)]\frac{\partial^2\hat{\phi}}{\partial z^2}+\frac{2}{\gamma q_0}\frac{\partial}{\partial t}\right.$$
$$\times\left[\sin[\gamma(\theta^*-\theta)]\frac{\partial\hat{\phi}}{\partial r}-\frac{\gamma}{r}\cos[\gamma(\theta^*-\theta)]\frac{\partial\hat{\phi}}{\partial\theta}\right]+\left[\left(\frac{1}{\gamma^2}+1\right)\sin^2[\gamma(\theta^*-\theta)]-1\right]\frac{\partial^2\hat{\phi}}{\partial r^2}$$
$$\left.-\frac{2}{\gamma r}\sin[\gamma(\theta^*-\theta)]\cos[\gamma(\theta^*-\theta)]\frac{\partial^2\hat{\phi}}{\partial r\,\partial\theta}\right\},$$

where we have put
$$\hat{\phi}=\phi/\cos[\gamma(\theta^*-\theta)]. \tag{3.20}$$

Hence, upon introducing the new independent variable

$$\eta\equiv\sin[\gamma(\theta^*-\theta)], \tag{3.21}$$

(2.30) becomes

$$r\left[\frac{1}{q_0^2}\frac{\partial^2\hat{\phi}}{\partial t^2}-(1-\eta^2)\frac{\partial^2\hat{\phi}}{\partial z^2}\right]+\frac{2}{\gamma q_0}\frac{\partial}{\partial t}\left[\eta r\frac{\partial\hat{\phi}}{\partial r}+\gamma^2(1-\eta^2)\frac{\partial\hat{\phi}}{\partial\eta}\right]+\left[\left(\frac{1}{\gamma^2}+1\right)\eta^2-1\right]r\frac{\partial^2\hat{\phi}}{\partial r^2}$$
$$+2\eta(1-\eta^2)\frac{\partial^2\hat{\phi}}{\partial r\,\partial\eta}=[H_0(\eta)+irH_1(\eta)]\exp\{i(k_3z-k_1U_\infty t)\}\exp\left\{ir\left[k_1\frac{U_\infty\Delta}{r}+\frac{k_2\Psi}{rU_\infty}\right]\right\}$$
$$\text{for}\quad \eta_0\leqslant\eta<1, \tag{3.22}$$

where we have put
$$\eta_0\equiv\sin[\gamma(\theta^*-\theta_\infty)] \tag{3.23}$$

and $\eta=1$ corresponds to the maximum permissible turning of the Prandtl–Meyer flow.

Since $\partial\Psi/\partial x_1=-(\rho_0/\rho_\infty)U_2$ and $\partial\Psi/\partial x_2=(\rho_0/\rho_\infty)U_1$ and since $U_2\to0$, $U_1\to U_\infty$ and $\rho_0\to\rho_\infty$ while (3.1), (3.2), (3.4), (3.8), (3.9) and (A 7)–(A 10) show that $U_\infty\,\partial\Delta/\partial x_i\to\delta_{1,i}$ as $\theta\to\theta_\infty$, it follows from (2.33), (3.11) and (3.12) that

$$\mathbf{u}^{(I)}\to\mathbf{u}_\infty(x-U_\infty t,y,z)\quad\text{as}\quad\theta\to\theta_\infty.$$

Hence it follows from (2.28), (3.20) and (3.23) that the tangential velocity components will be continuous across the leading Mach line $\theta=\theta_\infty$ only if

$$\hat{\phi}=0\quad\text{at}\quad\eta=\eta_0. \tag{3.24}$$

On the other hand, the tangential velocity will remain finite at $r=0$ only if

$$r^{-1}\partial[\hat{\phi}\cos\gamma(\theta^*-\theta)]/\partial\theta$$

remains finite and, consequently, only if $(1-\eta^2)^{\frac{1}{2}}\hat{\phi}\to$ constant as $r\to0$. Hence, in view of (3.24), we must require that

$$\hat{\phi}=0\quad\text{at}\quad r=0. \tag{3.25}$$

Since it is clear from (3.21)–(3.25) that the time and z dependence can enter the solution only through a factor of the form $\exp[i(k_3z-k_1U_\infty t)]$, $\hat{\phi}$ must satisfy a second-order partial differential equation in r and η. It is easy to show in the usual way that this equation is hyperbolic and its characteristics are $\eta=$ constant and $r(\eta(1-\eta^2)^\sigma)^{\frac{1}{2}}=$ constant, where $\sigma\equiv1/(2\gamma^2)$. Hence the boundary condition (3.24) is imposed on a characteristic curve while the boundary condition (3.25) is imposed on the degenerate characteristic of the other family that corresponds to the intersection of these characteristics. It therefore follows from the theory of hyperbolic equations that the boundary conditions (3.24) and (3.25) are just sufficient to determine uniquely the solution to (3.22) everywhere in the expansion fan.

Now (3.22) has variable coefficients that depend on both r and η and it cannot be solved by separation of variables in any co-ordinate system (Goldstein 1970). But since the variable coefficients are linear in r and since (3.22) involves only first derivatives with respect to η, it can be reduced to a first-order linear partial differential equation by taking its Laplace transform with respect to r. Such equations can always be solved by the method of characteristics. Thus, defining the reduced variable $\bar{\phi}$ by

$$\hat{\phi} = \exp(-ik_1 U_\infty t + ik_3 z)\,\bar{\phi}(r,\eta), \tag{3.26}$$

letting

$$g(s,\eta) = \int_0^\infty e^{-sr}\bar{\phi}(r,\eta)\,dr \tag{3.27}$$

be the Laplace transform of this quantity and taking the Laplace transform of both sides of (3.22), we obtain upon using (3.11), (3.12) and (3.25) and collecting terms

$$\left[(1-\eta^2)(s^2-k_3^2)+\left(\frac{k_1}{Q_0\gamma}w\right)^2\right]\frac{\partial g}{\partial s}+\frac{2ik_1}{Q_0}w(1-\eta^2)\frac{\partial g}{\partial \eta} = 2\left[\frac{ik_1 w}{\gamma^2 Q_0}\eta - s(1-\eta^2)\right]g$$
$$+\frac{H_0(\eta)}{s-i(k_1U_\infty\Delta/r+k_2\Psi'/U_\infty r)}+\frac{iH_1(\eta)}{[s-i(k_1U_\infty\Delta/r+k_2\Psi'/U_\infty r)]^2}, \tag{3.28}$$

where we have put

$$Q_0 \equiv \frac{q_0}{U_\infty} = \left(\frac{1+\gamma^2(M_\infty^2-1)}{M_\infty^2}\right)^{\frac{1}{2}} \tag{3.29}$$

and

$$w \equiv (\eta s Q_0/ik_1)-\gamma. \tag{3.30}$$

Taking the Laplace transform of (3.24) we obtain

$$g(s,\eta_0) = 0. \tag{3.31}$$

Equation (3.28) is a first-order linear partial differential equation whose characteristic equations are

$$\frac{ds}{(1-\eta^2)(s^2-k_3^2)+(k_1w/Q_0\gamma)^2} = \frac{Q_0\,d\eta}{2ik_1w(1-\eta^2)} = \frac{dg}{2[(ik_1w\eta/Q_0\gamma^2)-s(1-\eta^2)]g+G}, \tag{3.32}$$

where we have put

$$G \equiv \frac{\eta Q_0}{ik_1[w-\gamma Q_0\alpha_1(\eta)]-ik_2\gamma Q_0\alpha_2(\eta)}\left\{H_0(\eta)+\frac{i\eta Q_0 H_1(\eta)}{ik_1[w-\gamma Q_0\alpha_1(\eta)]-ik_2\gamma Q_0\alpha_2(\eta)}\right\}, \tag{3.33}$$

$$\alpha_1(\eta) \equiv \frac{\eta}{\gamma}\frac{U_\infty\Delta}{r}-\frac{1}{Q_0} = \frac{1}{Q_0}\left[\frac{\eta}{\gamma}(1-\eta^2)^\sigma\Delta_0-1\right], \tag{3.34}$$

$$\alpha_2(\eta) \equiv \frac{\eta\Psi'}{\gamma U_\infty r} = Q_0\frac{\eta}{\gamma}\frac{(1-\eta^2)^\sigma}{(1-\eta_0^2)^{1/(\kappa-1)}} \tag{3.35}$$

and

$$\Delta_0(\eta) = \frac{Q_0(M_\infty^2-1)^{\frac{1}{2}}}{M_\infty(1-\eta_0^2)^\sigma}+I(\eta)-I(\eta_0) \tag{3.36}$$

and used (3.8), (3.9), (3.11), (3.12), (3.21), (3.23) and (3.29) to obtain the last four equations.

Using w as a new variable in place of s, we find that the characteristic equation given by the first two terms of (3.32) is equivalent to

$$w\frac{dw}{d\eta} = \frac{(1-\eta^2)\gamma^2[(w+\gamma)(3w+\gamma)+\eta^2(Q_0 k_3/k_1)^2]-w^2\eta^2}{2\gamma^2(1-\eta^2)\eta}. \tag{3.37}$$

Integrating the remaining characteristic equation along the characteristic curve of (3.37) passing through the point (η, s), we find that the solution to the partial differential equation (3.28) that satisfies the initial condition (3.31) is given by

$$g(\eta) = -\frac{Q_0^2}{2k_1\eta(1-\eta^2)^\sigma}\int_{\eta_0}^{\eta}\left\{H_0(\tilde\eta)+\frac{Q_0\tilde\eta H_1(\tilde\eta)}{k_1[W(s,\eta|\tilde\eta)-\gamma Q_0\alpha_1(\tilde\eta)]-k_2\gamma Q_0\alpha_2(\tilde\eta)}\right\}$$
$$\times\frac{\tilde\eta^2(1-\tilde\eta^2)^\sigma\exp\left[-\gamma\int_{\tilde\eta}^{\eta}\frac{d\hat\eta}{\hat\eta W(s,\eta|\hat\eta)}\right]}{W(s,\eta|\tilde\eta)\{k_1[W(s,\eta|\tilde\eta)-\gamma Q_0\alpha_1(\tilde\eta)]-k_2\gamma Q_0\alpha_2(\tilde\eta)\}}\frac{d\tilde\eta}{(1-\tilde\eta^2)}, \tag{3.38}$$

where $W(s,\eta|\tilde\eta)$ denotes the solution of

$$W\frac{dW}{d\tilde\eta} = \frac{(1-\tilde\eta^2)\gamma^2[(W+\gamma)(3W+\gamma)+\tilde\eta^2(Q_0k_3/k_1)^2]-W^2\tilde\eta^2}{2\gamma^2(1-\tilde\eta^2)\tilde\eta} \tag{3.39}$$

that satisfies the boundary condition

$$W = w(\eta) \equiv (\eta s Q_0/ik_1)-\gamma \quad \text{at} \quad \tilde\eta = \eta. \tag{3.40}$$

Taking the inverse Laplace transform of (3.38) and using (3.20), (3.21) and (3.26), we find that the acceleration potential ϕ is given by

$$\phi = \exp\{-i(t-r/U_r)k_1U_\infty+ik_3z\}\frac{r(1-\eta^2)^{\frac{1}{2}}}{2}$$
$$\times\int_{\eta_0}^{\eta}\left[K_1(r,\eta|\tilde\eta)H_0(\tilde\eta)+K_2(r,\eta|\tilde\eta)\frac{Q_0}{k_1}\tilde\eta H_1(\tilde\eta)\right]\frac{d\tilde\eta}{\tilde\eta(1-\tilde\eta^2)} \quad \text{for} \quad \eta_0 \leqslant \eta < 1, \tag{3.41}$$

where

$$K_m = -\frac{1}{2\pi i}\frac{\tilde\eta^3}{r\eta}\left(\frac{1-\tilde\eta^2}{1-\eta^2}\right)^\sigma\frac{Q_0^2}{k_1^2}$$
$$+\int_{a-i\infty}^{a+i\infty}\frac{\exp\left[(s-iU_\infty k_1/U_r)r-\gamma\int_{\tilde\eta}^{\eta}\frac{d\hat\eta}{\hat\eta W(s,\eta|\hat\eta)}\right]}{W(s,\eta|\tilde\eta)[W(s,\eta|\tilde\eta)-\gamma Q_0\alpha_1(\tilde\eta)-k_2\gamma Q_0\alpha_2(\tilde\eta)/k_1]^m}ds \quad \text{for} \quad m = 1,2 \tag{3.42}$$

and the constant a must be so chosen that the path of integration is to the right of the singularities of the integrand.

This completes the formal solution of the problem. The velocity and pressure fields can be calculated at all points of the expansion fan by inserting (3.16) and (3.41) in (2.26) and (2.28). It is evident from (3.16), (3.18), (3.19) and (3.41) that these quantities must be of the form

$$u_\nu = \exp\{-ik_1U_\infty(t-r/U_r)+ik_3z\}M_{\nu,j}(r,\theta)A_j \quad \text{for} \quad \nu = r,\theta,z, \tag{3.43}$$

$$p'/\rho_0 = \exp\{-ik_1U_\infty(t-r/U_r)+ik_3z\}N_j(r,\theta)A_j, \tag{3.44}$$

where the summation over the repeated index is to be from 1 to 3. The coupling coefficients $M_{\nu,j}$ and N_j are independent of the amplitudes A_j of the incident velocity field and can be calculated from (3.16), (3.18), (3.19) and (3.41). In fact, since it follows from (3.11), (3.12), (3.21), (3.34), (3.35) and (A 9) that

$$\frac{U_\infty}{r}\frac{\partial\Delta}{\partial\theta} = \frac{\alpha_1}{(1-\eta^2)^{\frac{1}{2}}}, \quad \frac{1}{U_\infty r}\frac{\partial\Psi}{\partial\theta} = \frac{\alpha_2}{(1-\eta^2)^{\frac{1}{2}}},$$

$$\frac{U_\infty\Delta}{r} = \frac{\beta_1}{(1-\eta^2)^{\frac{1}{2}}}, \quad \frac{\Psi}{U_\infty r} = \frac{\beta_2}{(1-\eta^2)^{\frac{1}{2}}},$$

where
$$\beta_1 = \frac{(1-\eta^2)^{\kappa/(\kappa-1)}\Delta_0(\eta)}{Q_0}, \quad \beta_2 = \frac{(1-\eta^2)^{\kappa/(\kappa-1)}}{(1-\eta_0^2)^{1/(\kappa-1)}}Q_0, \tag{3.45}$$

they can be written as
$$M_{\nu,j} = M^{(0)}_{\nu,j} + m_{\nu,j}, \tag{3.46}$$

$$N_j = \frac{k_1 U_\infty}{k_3} M^{(0)}_{z,j} - U_r M^{(0)}_{r,j} - U_\theta\left(M_{\theta,j} - \frac{\alpha_j}{(1-\eta^2)^{\frac{1}{2}}} e^{ir\mu}\right), \tag{3.47}$$

where

$$M^{(0)}_{\nu,j} \equiv \frac{2(1-\eta^2)^{\frac{1}{2}}}{\gamma(\kappa+1)}\int_{\eta_0}^{\eta} K^{(1)}_\nu(r,\eta\,|\,\tilde\eta)\frac{\alpha_j(\tilde\eta)}{(1-\tilde\eta^2)^{\frac{3}{2}}}\,d\tilde\eta + \frac{Q_0(1-\eta^2)^{\frac{1}{2}}k_l}{2k_1}\int_{\eta_0}^{\eta} K^{(2)}_\nu(r,\eta\,|\,\tilde\eta)\,D_{l,j}(\tilde\eta)\frac{d\tilde\eta}{(1-\tilde\eta^2)^{\frac{3}{2}}}, \tag{3.48}$$

$$m_{r,j} = \frac{\beta_j(\eta)}{(1-\eta^2)^{\frac{1}{2}}} e^{ir\mu(\eta)}, \tag{3.49a}$$

$$m_{\theta,j} = \left[e^{ir\mu(\eta)} - \frac{2i}{\kappa+1}\frac{1-e^{ir\mu(\eta)}}{r\mu(\eta)}\right]\frac{\alpha_j(\eta)}{(1-\eta^2)^{\frac{1}{2}}} + \frac{i\gamma}{2}\frac{1-[1-ir\mu(\eta)]\,e^{ir\mu(\eta)}}{r\mu^2(\eta)}\frac{k_l D_{l,j}(\eta)}{\eta(1-\eta^2)^{\frac{1}{2}}}, \tag{3.49b}$$

$$m_{z,j} = \delta_{j,3}\, e^{ir\mu(\eta)}, \tag{3.49c}$$

$$\mu(\eta) \equiv (\gamma/\eta)\,[k_1\alpha_1(\eta) + k_2\alpha_2(\eta)], \tag{3.50}$$

$$K^{(m)}_r = \left(1 + \frac{ik_1 r}{U_r}U_\infty\right)K_m + r\frac{\partial K_m}{\partial r} \tag{3.51a}$$

$$K^{(m)}_\theta = \frac{\gamma}{(1-\eta^2)^{\frac{1}{2}}}\left[\left(\eta + \frac{1-\eta^2}{\eta}\frac{ik_1 rU_\infty}{U_r}\right)K_m - (1-\eta^2)\frac{\partial K_m}{\partial\eta}\right] \quad \text{for} \quad m = 1, 2, \tag{3.51b}$$

$$K^{(m)}_z = ik_3 r K_m \tag{3.51c}$$

$$D_{l,j} = \alpha_l(\eta)\,\alpha_j(\eta) + \beta_l(\eta)\,\beta_j(\eta) \quad \text{for} \quad l,j = 1,2, \tag{3.52a}$$

$$D_{3,j} = D_{j,3} = \delta_{j,3}(1-\eta^2) \tag{3.52b}$$

and we have set $\alpha_3 = \beta_3 = 0$.

Equation (3.39) is an Abel equation of the second kind that cannot be solved in closed form. There is, therefore, no hope of evaluating the contour integrals in (3.42) analytically. However, in appendix B we expand the solution to the initial-value problems (3.39) and (3.40) in inverse powers of w and in appendix C we use this result to obtain an expansion for K_m in ascending powers of $k_1 r$. Relatively simple formulae are obtained for the case where k_2/k_1 and k_3/k_1 are finite. A more complicated result is given for the case where $k_2 r$ is arbitrary.

4. Supersonic turbulent flow around a corner

With the recent advances in laser-Doppler velocimetry, it is now possible to measure turbulence velocity correlations and their spectra in supersonic flows. An interesting experiment might therefore consist of placing a wedge at an angle of attack to a turbulent supersonic stream and measuring the turbulence spectra in the centred expansion fan at the leading edge of the suction surface. The configuration is the same as the one shown in figure 2(*c*).

It should be possible to obtain a relatively homogeneous turbulence upstream of the wedge. The results obtained in the previous section can then be used to analyse the flow. The conditions for the validity of the basic model should be pretty much the same as those deduced by Hunt (1973) when the mean-flow Mach number is not too large. See Hunt (1977) for a discussion of this point.

We assume that the turbulence upstream of the wedge is homogeneous and that its three-dimensional spectrum is known. The spectra and covariances of the turbulence in the expansion fan are related to this spectrum through the coupling coefficients deduced in the previous section. The relations are formally the same as those given by Hunt (1973). Thus the one-dimensional spectrum

$$\Theta_{\nu,\mu}(\mathbf{x},k_1) = \frac{U_\infty}{2\pi}\int_{-\infty}^{\infty} R_{\nu,\mu}(\mathbf{x},\tau)\exp(ik_1 U_\infty \tau)\,d\tau, \quad \nu,\mu = r,\theta,z, \tag{4.1}$$

where

$$R_{\nu,\mu}(\mathbf{x},\tau) = \overline{u_\nu(\mathbf{x},t)\,u_\mu(\mathbf{x},t+\tau)} \tag{4.2}$$

is the one-point turbulence velocity correlation tensor (and the overbar denotes a time average), is related to the three-dimensional upstream turbulence spectrum $\Phi^{(\infty)}_{i,j}(k_1,k_2,k_3)$ via the relation

$$\Theta_{\nu,\mu}(\mathbf{x},k_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M^*_{\nu,j}\,M_{\mu,n}\,\Phi^{(\infty)}_{j,n}(\mathbf{k})\,dk_2\,dk_3, \tag{4.3}$$

where the asterisk denotes the complex conjugate.

As is usual in problems involving the interaction of turbulence with solid obstacles, we assume that the upstream turbulence is isotropic, so that (Batchelor 1953, p. 49)

$$\Phi^{(\infty)}_{j,n} = (E(k)/4\pi k^4)\,(k^2\delta_{j,n} - k_j k_n), \tag{4.4}$$

where $k = |\mathbf{k}|$ and $E(k)$ is the energy spectrum function. The one-dimensional spectrum for the axial velocity upstream of the wedge, which is given by

$$\Theta^{(\infty)}_{1,1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{u_{\infty,1}(\mathbf{x}-\hat{\mathbf{i}}U_\infty t)\,u_{\infty,1}(\mathbf{x}-\hat{\mathbf{i}}U_\infty(t+\tau))}\exp(ik_1 U_\infty\tau)\,d\tau, \tag{4.5}$$

is related to $E(k)$ by (Batchelor 1953, p. 50)

$$E(k) = k^3\frac{d}{dk}\left(\frac{1}{k}\frac{d}{dk}\,\Theta^{(\infty)}_{1,1}(k)\right). \tag{4.6}$$

A convenient choice for $\Theta^{(\infty)}_{1,1}$ which is in good agreement with experiment is the von Kármán spectrum

$$\Theta^{(\infty)}_{1,1}(k_1) = c_1/(g_2 + k_1^2 l^2)^{\frac{5}{6}}, \tag{4.7}$$

where $c_1 = g_1 l \overline{u^2_{\infty,1}}$, $g_1 \simeq 0{\cdot}1955$, $g_2 \simeq 0{\cdot}558$, and l denotes the integral scale of the turbulence. Then it follows from (4.4) and (4.6) that

$$\Phi^{(\infty)}_{j,n} = \frac{55}{36\pi l^{\frac{2}{3}}} \frac{g_1 \overline{u^2_{\infty,1}}[k^2\delta_{j,n} - k_j k_n]}{(g_2/l^2 + k^2)^{\frac{17}{6}}} \tag{4.8}$$

and that
$$\Theta^{(\infty)}_{2,2} = \frac{U_\infty}{2\pi}\int_{-\infty}^{\infty} \overline{u_{\infty,2}(\mathbf{x} - \hat{\mathbf{i}}U_\infty t)\, u_{\infty,2}(\mathbf{x} - \hat{\mathbf{i}}U_\infty(t+\tau))} \exp(ik_1 U_\infty \tau)\, d\tau \tag{4.9}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi^{(\infty)}_{2,2}\, dk_2\, dk_3 = \tfrac{1}{2}\Theta^{(\infty)}_{1,1}\left[1 + \frac{5}{3}\frac{k_1^2 l^2}{g_2 + (k_1 l)^2}\right]. \tag{4.10}$$

We can now calculate $\Theta_{\nu,\mu}$ at all points of the expansion fan by substituting (3.46) [with the M's defined by (3.48)–(3.52)] and (4.8) into (4.3). The coupling coefficients ultimately depend on K_m, which must be evaluated from the contour integral (3.42). But the dominant contribution to the integral in (4.3) comes from the region where $k = O(l^{-1})$, when $k_1 l = O(1)$, while $\Phi^{(\infty)}_{j,k}$ will be small when $k_1 l \gg 1$. Hence we ought to be able to use (C 5) at all points where $r/l \ll 1$. But when this expansion is substituted into (4.3) via (3.46), (3.48) and (3.51) the resulting integral will diverge because $\Phi^{(\infty)}_{j,k}$ does not approach zero at a fast enough rate as $k_2 \to \infty$. This difficulty can be overcome by using (C 9) and (C 10), which do not require that $k_2 r$ be small. On the other hand the use of (C 5) will not lead to divergent integrals when the $O(k_1^2 r^2)$ terms are omitted. The results will then agree with those obtained from (C 9) and (C 10).

Making the substitutions alluded to above and carrying out the integrations over k_2 and k_3, we obtain

$$\Theta_{r,r} = \Theta^{(\infty)}_{1,1} Q^2_{r,1} + \Theta^{(\infty)}_{2,2} Q^2_{r,2} + g_{r,r}(\eta)\,(k_1 r)^{\frac{5}{3}} + O(k_1^2 r^2), \tag{4.11}$$

$$\Theta_{\theta,\theta} = \Theta^{(\infty)}_{1,1} Q^2_{\theta,1} + \Theta^{(\infty)}_{2,2} Q^2_{\theta,2} + g_{\theta,\theta}(\eta)\,(k_1 r)^{\frac{5}{3}} + O(k_1^2 r^2), \tag{4.12}$$

$$\begin{aligned}\Theta_{r,\theta} = {} & \Theta^{(\infty)}_{1,1}\{Q_{\theta,1}Q_{r,1} + i(rk_1)\,[Q_{r,1}q^{(1)}_{\theta,1} - Q_{\theta,1}q^{(1)}_{r,1} - \tfrac{1}{2}(Q_{r,1}q^{(2)}_{\theta,2} - Q_{\theta,1}q^{(2)}_{r,2} \\ & + Q_{r,2}q^{(2)}_{\theta,1} - Q_{\theta,2}q^{(2)}_{r,1} + Q_{r,1}q^{(3)}_{\theta,3} - Q_{\theta,1}q^{(3)}_{r,3})]\} + \Theta^{(\infty)}_{2,2}[Q_{r,2}Q_{\theta,2} + i(rk_1) \\ & \times (Q_{r,2}q^{(1)}_{\theta,2} - Q_{\theta,2}q^{(1)}_{r,2})] + g_{r,\theta}(\eta)\,(k_1 r)^{\frac{5}{3}} + O(k_1^2 r^2),\end{aligned} \tag{4.13}$$

where
$$Q_{r,j} = \frac{2(1-\eta^2)^{\frac{1}{2}}}{\gamma}\hat{Q}_j(\eta) + \frac{\beta_j(\eta)}{(1-\eta^2)^{\frac{1}{2}}}, \tag{4.14}$$

$$q^{(l)}_{r,j} = (1-\eta^2)^{\frac{1}{2}}\,\hat{q}_{l,j}(\eta) + \frac{2\delta_{1,l}(1-\eta^2)^{\frac{1}{2}}}{Q_0\eta}\hat{Q}_j(\eta) + \frac{\gamma\alpha_l(\eta)\beta_j(\eta)}{\eta(1-\eta^2)^{\frac{1}{2}}}, \tag{4.15}$$

$$Q_{\theta,j} = 2\eta\hat{Q}_j(\eta) + \frac{\kappa-1}{\kappa+1}\frac{\alpha_j(\eta)}{(1-\eta^2)^{\frac{1}{2}}}, \tag{4.16}$$

$$q^{(l)}_{\theta,j} = \nu(\eta)\,\hat{q}_{l,j}(\eta) - \frac{\gamma\delta_{1,l}(1-\eta^2)}{Q_0\eta^2}\hat{Q}_j(\eta) + \frac{\gamma}{\eta(1-\eta^2)^{\frac{1}{2}}}\left[\frac{\kappa}{\kappa+1}\alpha_l(\eta)\,\alpha_j(\eta) - \tfrac{1}{4}D_{l,j}(\eta)\right], \tag{4.17}$$

$$\hat{Q}_j(\eta) \equiv \frac{1}{\kappa+1}\int_{\eta_0}^{\eta}\frac{\alpha_j(\tilde{\eta})}{(1-\tilde{\eta}^2)^{\frac{3}{2}}}\,d\tilde{\eta}, \tag{4.18}$$

$$\hat{q}_{l,j}(\eta) \equiv \int_{\eta_0}^{\eta}\frac{\tau(\eta)}{\tilde{\eta}\tau(\tilde{\eta})}\left[\frac{2\alpha_l(\tilde{\eta})\,\alpha_j(\tilde{\eta})}{\kappa+1} + \tfrac{1}{2}D_{l,j}(\tilde{\eta})\right]\frac{d\tilde{\eta}}{(1-\tilde{\eta}^2)^{\frac{3}{2}}} - \frac{6\delta_{1,l}\hat{\tau}(\eta)}{(\kappa+1)\,Q_0}\int_{\eta_0}^{\eta} J(\tilde{\eta}|\eta)\,\alpha_j(\tilde{\eta})\frac{d\tilde{\eta}}{(1-\tilde{\eta}^2)^{\frac{3}{2}}}, \tag{4.19}$$

$$\nu(\eta) = \frac{\gamma}{2}\left[\eta - \frac{\gamma^2(1-\eta^2) - \eta^2}{2\gamma^2\eta}\right] \tag{4.20}$$

and $\hat{\tau}(\eta)$ is defined by (C 12).

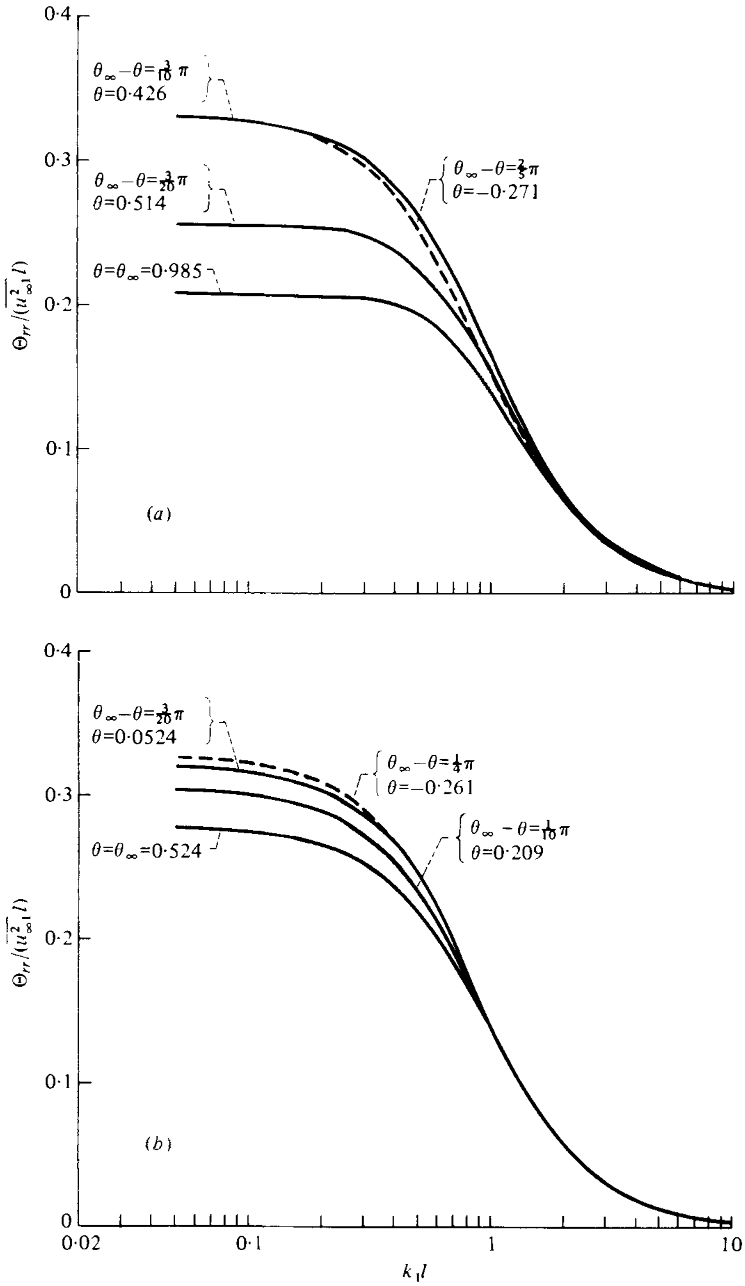

FIGURE 4. Radial velocity spectra. (*a*) $M_\infty = 1{\cdot}2$. (*b*) $M_\infty = 2{\cdot}0$.

The expressions for the g's are even more complicated than those for the q's and will therefore not be given here. It is worth noting however that the spectral density functions $\Theta_{r,r}$ and $\Theta_{\theta,\theta}$ vary with r like $(k_1 r)^{\frac{5}{3}}$ rather than having the $(k_1 r)^2$ dependence that one might anticipate from the expansion (C 5). This behaviour results from the integration with respect to k_2 of the exponential factors that appear in (C 9) and

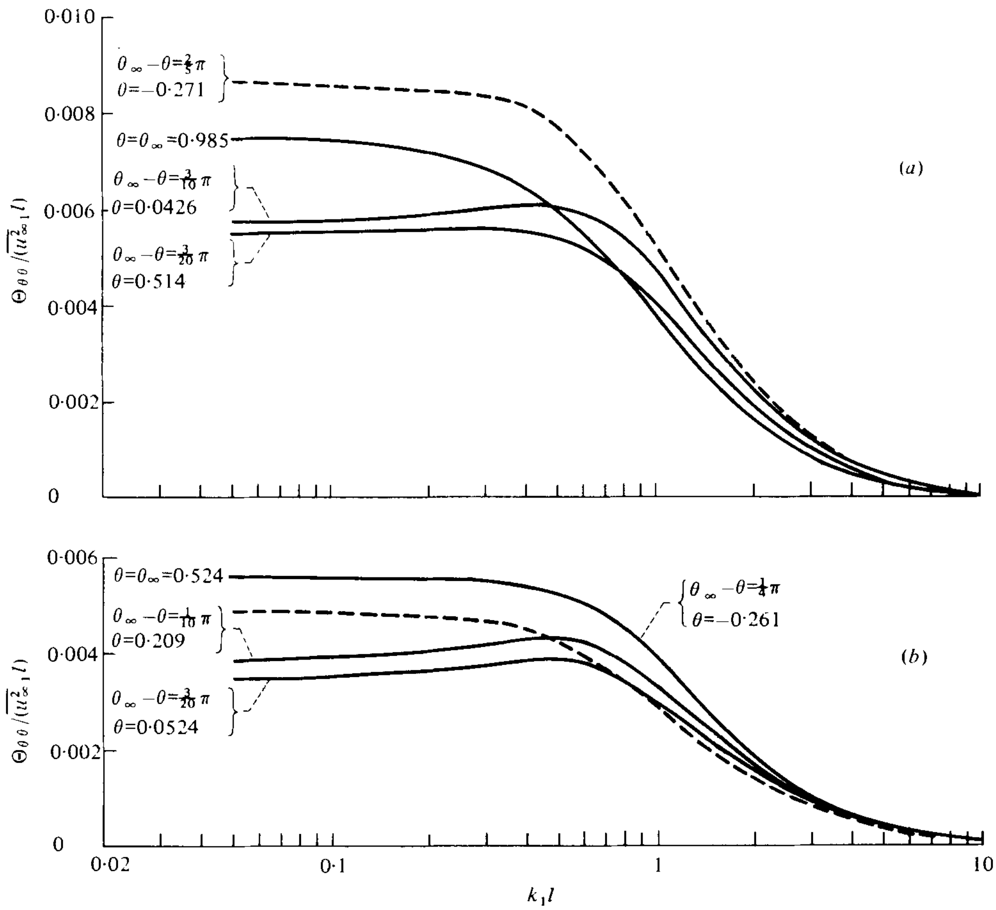

FIGURE 5. Circumferential velocity spectra. (a) $M_\infty = 1·2$. (b) $M_\infty = 2·0$.

(C 10). When they are expanded for small $k_2 r$ to obtain (C 5) the integrals in (4.3) become divergent.

Figures 4 and 5 are plots of the one-dimensional autocorrelation spectra for the radial and circumferential velocities at several circumferential locations in the expansion fan, the dashed curves corresponding to negative θ. They were calculated from (4.11) and (4.12) with the $O((k_1 r)^{\frac{2}{3}})$ terms omitted and are therefore independent of the radial co-ordinate r.

It might at first appear that the vanishing of the normal velocity on the surface of the wedge would require Θ_{rr} to go to zero at $r = 0$. But the expansion fan meets the wedge at a single point which could just as well be associated with the unbounded upstream region as with the bounded region downstream of the expansion fan. Of course, the normal surface velocity must vanish in this downstream region and as θ varies through the expansion fan the radial surface velocity varies from its non-zero upstream value to a value that will allow the downstream surface velocity to vanish. But since the tangential velocity is continuous across the Mach line between the expansion fan and the downstream region it is easy to see that the normal surface velocity could not in general be made to vanish in the latter region if the radial velocity were zero in the expansion fan.

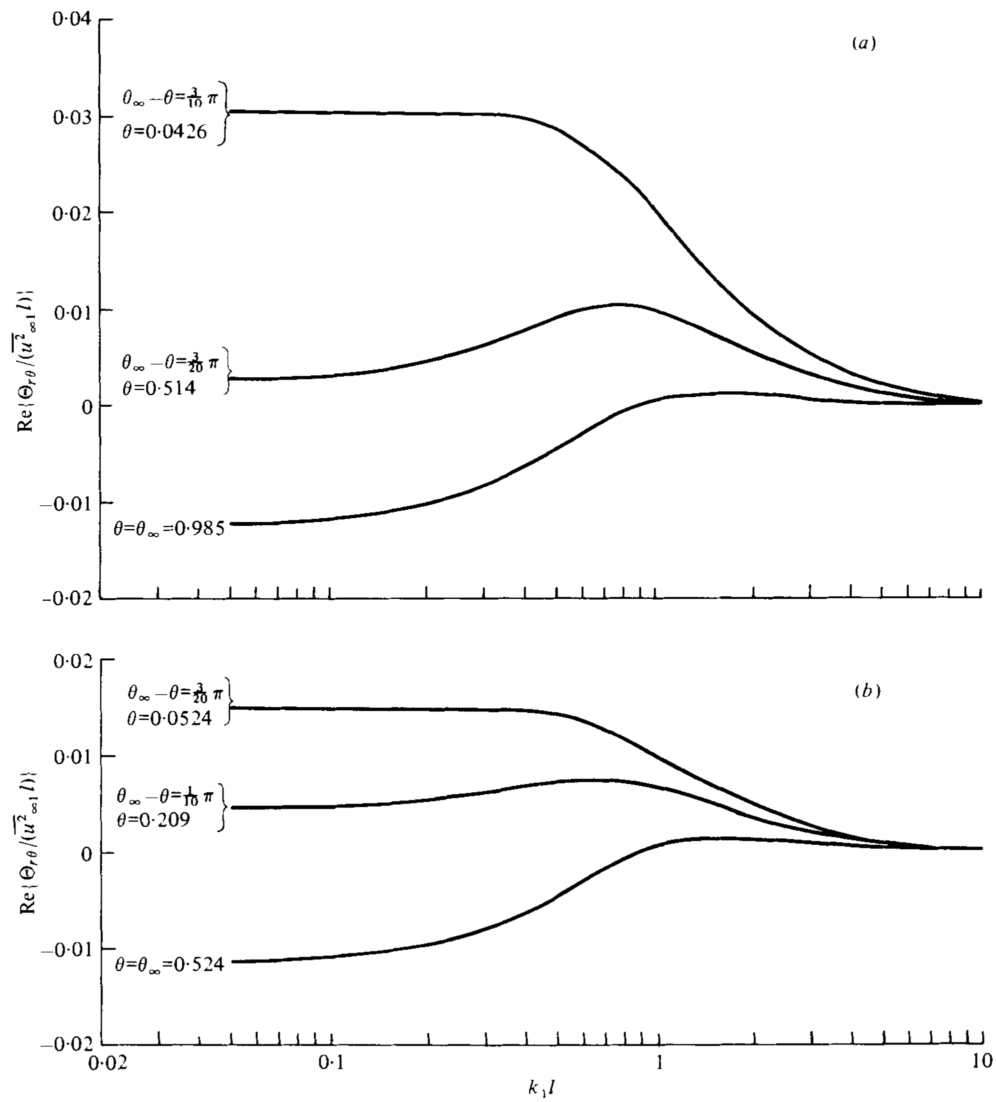

FIGURE 6. Real part of radial–circumferential velocity cross-spectra. (*a*) $M_\infty = 1{\cdot}2$. (*b*) $M_\infty = 2{\cdot}0$.

Since the tangential velocities are always continuous across the Mach waves, the $\Theta_{r,r}$ correlations at the upstream edge of the expansion fan, i.e. at $\theta = \theta_\infty$, are identical with the r, r upstream autocorrelation spectra. On the other hand, the discontinuity in the circumferential velocities across the initial Mach wave causes $\Theta_{\theta,\theta}$ to differ from the corresponding upstream spectra. However, it turns out that they have the same shape as the upstream spectra and differ only by a multiplicative factor of $\gamma^2 = \frac{1}{6}$ (for air).

The curves show that the overall levels of the radial spectra always increase with increasing $\theta_\infty - \theta$, especially at the lower frequencies. On the other hand, the low frequency portions of the circumferential velocity spectra first decrease and then increase with increasing $\theta_\infty - \theta$. At the lower Mach numbers the initial decrease is

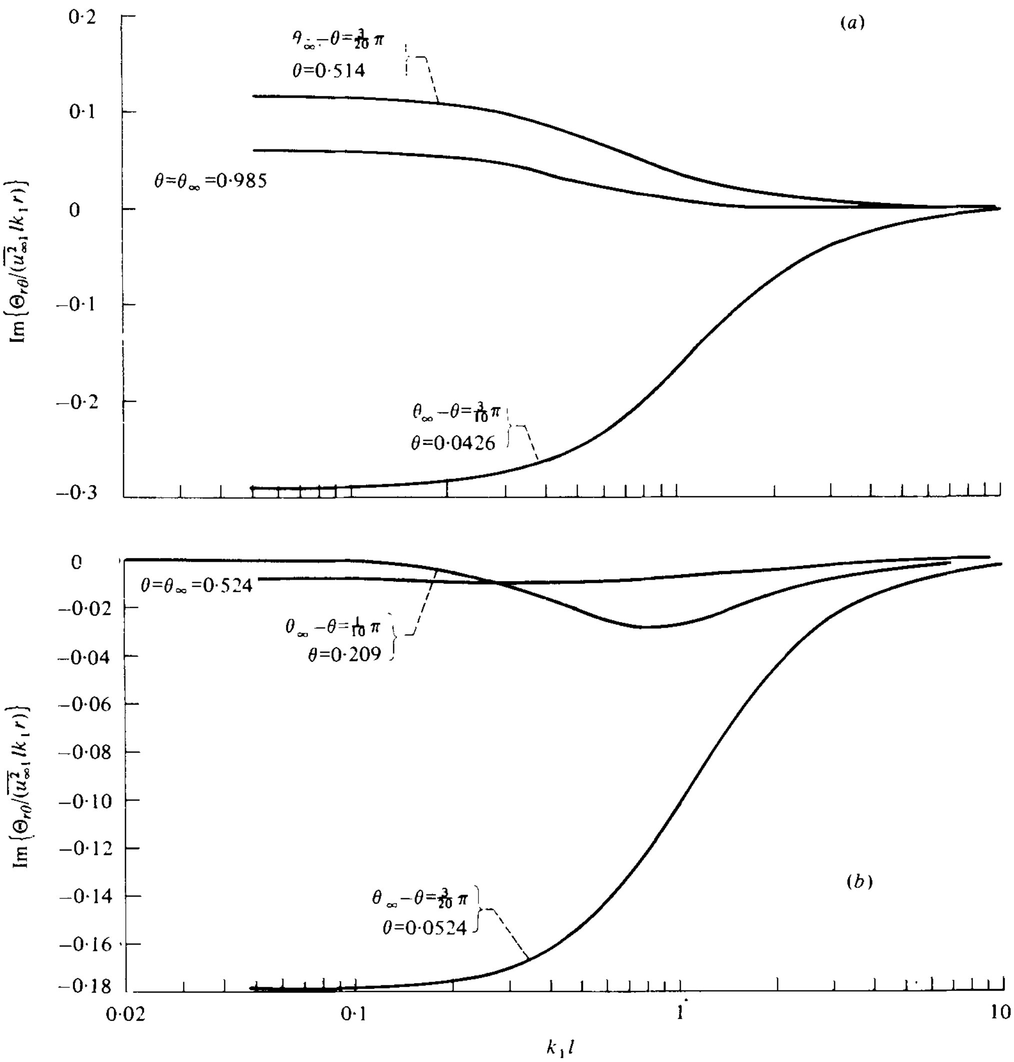

FIGURE 7. Imaginary part of radial–circumferential velocity cross-spectra. (*a*) $M_\infty = 1{\cdot}2$. (*b*) $M_\infty = 2{\cdot}0$.

compensated for by a corresponding increase in the high frequency portion of the spectrum, so that the net effect is a shift to higher frequencies followed by an overall increase in level. At the higher Mach numbers the final increase is compensated for by a decrease in the higher frequency components. The net effect is therefore an overall decrease in level with a shift to lower frequencies at the larger turning angles.

At small values of θ the circumferential spectra exhibit a slight hump near $k_1 l = 0{\cdot}7$ and, as can easily be anticipated from (4.7) and (4.10)–(4.12), the radial and circumferential spectra both decay like $k_1^{-\frac{5}{3}}$ for large $k_1 l$. Even though the results for the two Mach numbers are plotted at different angular locations, it can be seen by interpolation that the variation with Mach number is rather complex.

The radial–circumferential velocity cross-spectra $\Theta_{r,\theta}$ are shown in figures 6–8. These results were calculated from (4.13) with $O((k_1 r)^{\frac{5}{3}})$ terms omitted. The real and

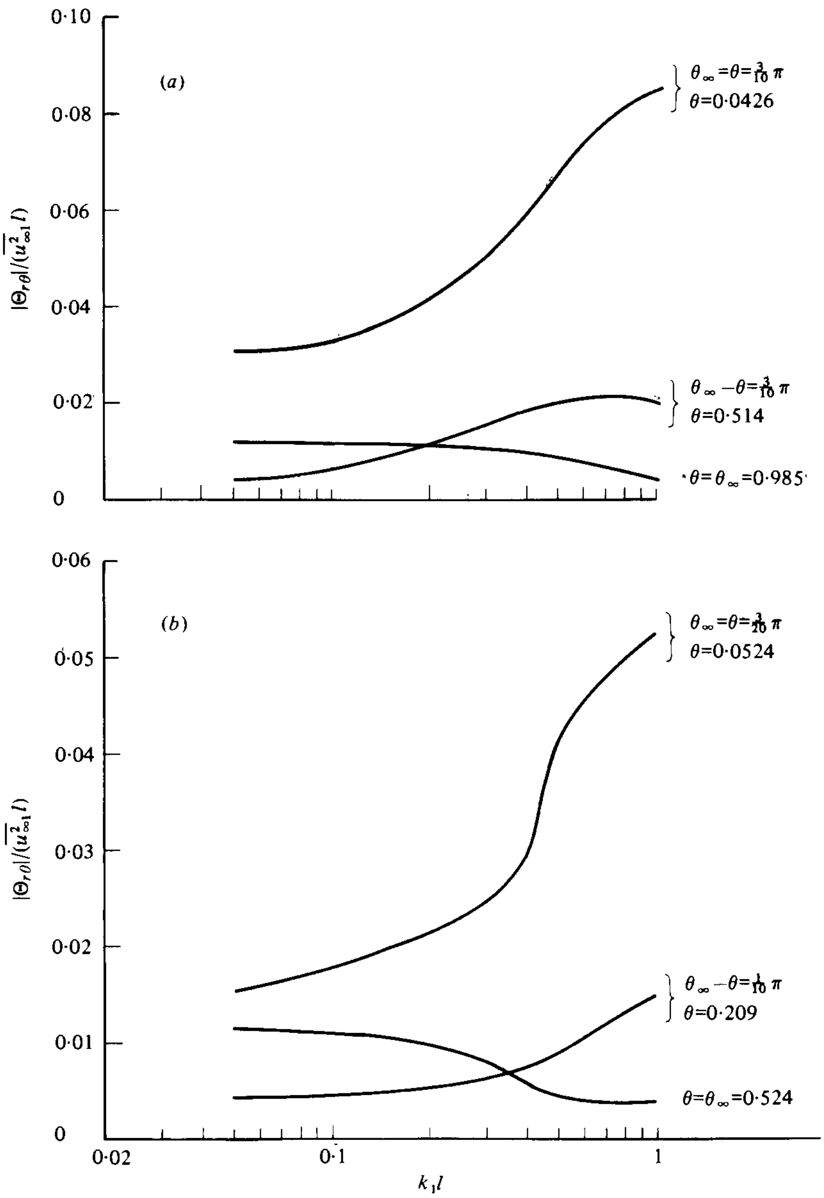

FIGURE 8. Amplitude of radial–circumferential velocity cross-spectra at $r/l = 0{\cdot}5$. (*a*) $M_\infty = 1{\cdot}2$. (*b*) $M_\infty = 2{\cdot}0$.

imaginary parts are plotted in figures 6 and 7, respectively. The former are independent of radial position and the latter, which vary linearly with this quantity, are normalized by $k_1 r$.

Since the incident turbulence is homogeneous and isotropic, the imaginary part of $\Theta_{r,\theta}$ will vanish upstream of the expansion fan.† It will not, however, vanish along

† Notice that, even though $\int_{-\infty}^{\infty} \Theta_{r,\theta}\, dk_1 = 0$ in the upstream region, $\Theta_{r\theta}$ itself will not vanish unless $\theta = 0$ or $\frac{1}{2}\pi$.

the leading Mach wave because of the discontinuity in the circumferential velocities across this line.

It is worth noting that the real parts of the cross-spectra exhibit a hump that disappears at the larger values of $\theta_\infty - \theta$. On the other hand, the results show that the imaginary parts of the spectra can be an order of magnitude larger than the real parts as $k_1 r$ approaches unity.

Figure 8 is a plot of the magnitudes of the cross-spectra at $r = 0{\cdot}5l$. Its range is restricted to $k_1 l < 1$ in order to ensure that $k_1 r$ remains less than unity. The curves therefore give no information about the high frequency fall-off of the spectra. The positive slope exhibited by the intermediate positive θ curves is due to the $k_1 r$ factor that multiplies the large imaginary parts of the $\Theta_{r,\theta}$.

The author would like to thank Oliver Reese and Dr Lynn Albers for carrying out the numerical computations. Thanks are also due to Gita Lynn for checking the manuscript, to Prof. J. E. Ffowcs Williams for his helpful suggestions, to Dr J. C. R. Hunt for sending me a preliminary copy of his paper and especially to his student Mr Paul Durbin for pointing out the transformation (2.17), which helped to simplify the analysis.

Appendix A

In order to evaluate $\mathbf{u}^{(I)}$ and the terms that enter the right side of (2.30) through $\mathbf{u}^{(I)}$ we must calculate the drift function Δ defined by (2.22). But since equations (2.20) imply that $dx/U_x = d\mathscr{S}/|\mathbf{U}|$, where $\mathscr{S}$ denotes the arc length along a streamline, and since $U_x = U_\infty$ in the region upstream of the expansion fan, this becomes

$$\Delta = \frac{r_0 \cos\theta_\infty}{U_\infty} + \int_0^{\mathscr{S}(r,\theta)} \frac{d\mathscr{S}}{|\mathbf{U}|} \quad \text{for} \quad \theta > \theta_\infty, \tag{A 1}$$

where

$$\theta_\infty = \sin^{-1} M_\infty^{-1} \tag{A 2}$$

is the upstream Mach angle and r_0 is the radius at which the streamline passing through (r, θ) enters the expansion fan. Since Ψ is constant along the streamlines, it follows from (3.11) that

$$r_0/r = [\cos\gamma(\theta^* - \theta)/\cos\gamma(\theta^* - \theta_\infty)]^{1/\gamma^2}, \tag{A 3}$$

which also shows that the polar co-ordinates (r_s, θ_s) of any point on the streamline that passes through (r, θ) are related by

$$r_s/r = [\cos\gamma(\theta^* - \theta)/\cos\gamma(\theta^* - \theta_s)]^{1/\gamma^2}. \tag{A 4}$$

Then since

$$d\mathscr{S} = -r_s \left[\left(\frac{1}{r_s}\frac{dr_s}{d\theta_s}\right)^2 + 1\right]^{\frac{1}{2}} d\theta_s,$$

it follows that

$$d\mathscr{S} = -r_s \left[\left(\frac{\tan\gamma(\theta^* - \theta_s)}{\gamma}\right)^2 + 1\right]^{\frac{1}{2}} d\theta_s. \tag{A 5}$$

Also, since (3.1) and (3.2) show that

$$|\mathbf{U}| = q_0 \left[\left(\frac{\tan\gamma(\theta^* - \theta_s)}{\gamma}\right)^2 + 1\right]^{\frac{1}{2}} \cos\gamma(\theta^* - \theta_s),$$

it follows from (3.11), (A 1) and (A 3)–(A 5) that

$$\Delta = \frac{\Psi}{q_0^2}[\cos\gamma(\theta^* - \theta_\infty)]^{1/\gamma^2-1}\left[\frac{(q_0/U_\infty)\cos\theta_\infty}{[\cos\gamma(\theta^*-\theta_\infty)]^{1/\gamma^2}} - \int_{\theta_\infty}^{\theta}\frac{d\theta_s}{[\cos\gamma(\theta^*-\theta_s)]^{1/\gamma^2+1}}\right]. \quad \text{(A 6)}$$

Upon taking $V = \tan\gamma(\theta^* - \theta_s)$ as the variable of integration and using (3.7), we find that the integral can be written as

$$-\frac{1}{\gamma}\int_{\tan\gamma(\theta^*-\theta_\infty)}^{\tan\gamma(\theta^*-\theta)}(1+V^2)^{1/(k-1)}\,dV,$$

which can be expressed in terms of hypergeometric functions to obtain

$$\Delta = \frac{\Psi}{q_0^2}[\cos\gamma(\theta^*-\theta_\infty)]^{1/\gamma^2-1}\left[\frac{(q_0/U_\infty)\cos\theta_\infty}{[\cos\gamma(\theta^*-\theta_\infty)]^{1/\gamma^2}} + I(\sin\gamma(\theta^*-\theta)) - I(\sin\gamma(\theta^*-\theta_\infty))\right], \quad \text{(A 7)}$$

where

$$I(Z) \equiv \frac{1}{\gamma}\frac{Z}{(1-Z^2)^{\frac{1}{2}}}F\left(-\frac{1}{\kappa-1}, \frac{1}{2}; \frac{3}{2}; -\frac{Z^2}{1-Z^2}\right) = \frac{Z}{\gamma}F\left(\frac{1}{2}, 1+\frac{1}{2\gamma^2}; \frac{3}{2}; Z^2\right) \quad \text{(A 8)}$$

and F denotes the hypergeometric function in the usual notation.

It follows from (3.1), (3.2), (3.4), (3.10) and (A 6) that

$$\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \frac{(\Delta/r)\,U_r - 1}{c_0}, \quad \text{(A 9)}$$

and since (3.11) and (A 8) show that $\Delta \propto r$,

$$\partial\Delta/\partial r = \Delta/r. \quad \text{(A 10)}$$

Appendix B

In this appendix we obtain an asymptotic solution to the initial-value problems (3.39) and (3.40) that is valid for large values of w and, consequently, of s. To this end, we write (3.39) in the form

$$\frac{dW^2}{d\tilde\eta} + \left[\frac{\tilde\eta^2 - 3\gamma^2(1-\tilde\eta^2)}{\gamma^2(1-\tilde\eta^2)\,\tilde\eta}\right]W^2 = \frac{4\gamma W + \gamma^2 + (Q_0 k_3/k_1)^2\tilde\eta^2}{\tilde\eta}$$

and treat the right side of this result as a known source term. The equation is then linear in W^2 and can be solved by standard methods. The solution that satisfies the boundary condition (3.40) is

$$W = w\left(\frac{\tilde\eta}{\eta}\right)^{\frac{3}{2}}\left(\frac{1-\tilde\eta^2}{1-\eta^2}\right)^{\frac{1}{2}\sigma}\left\{1 + \frac{\eta^3(1-\eta^2)^\sigma}{w^2}\int_\eta^{\tilde\eta}\frac{[4\gamma W(s,\eta|\hat\eta) + \gamma^2 + (Q_0 k_3/k_1)^2\hat\eta^2]}{\hat\eta^4(1-\hat\eta^2)^\sigma}\,d\hat\eta\right\}^{\frac{1}{2}},$$

where $\sigma = 1/(2\gamma^2)$. For large values of w the integral can be neglected and the solution is given by

$$w\left(\frac{\tilde\eta}{\eta}\right)^{\frac{3}{2}}\left(\frac{1-\tilde\eta^2}{1-\eta^2}\right)^{\frac{1}{2}\sigma}.$$

We can attempt to proceed iteratively by using this for the value of W in the integral. Making this substitution and expanding for large w, we obtain

$$\frac{W}{\tilde{\eta}^{\frac{3}{2}}(1-\tilde{\eta}^2)^{\frac{1}{2}\sigma}} = \frac{w}{\eta^{\frac{3}{2}}(1-\eta^2)^{\frac{1}{2}\sigma}} + 2\gamma J(\tilde{\eta}|\eta) + O(w^{-1}),$$

where
$$J(\tilde{\eta}|\eta) \equiv \int_{\eta}^{\tilde{\eta}} \frac{d\hat{\eta}}{\hat{\eta}^{\frac{5}{2}}(1-\hat{\eta}^2)^{\frac{1}{2}\sigma}}. \quad \text{(B 1)}$$

Using this for the next iteration, we obtain after some simplification of the results

$$\frac{W}{\tilde{\eta}^{\frac{3}{2}}(1-\tilde{\eta}^2)^{\frac{1}{2}\sigma}} = \frac{w}{\eta^{\frac{3}{2}}(1-\eta^2)^{\frac{1}{2}\sigma}} + 2\gamma J(\tilde{\eta}|\eta) + \frac{\eta^{\frac{3}{2}}(1-\eta^2)^{\frac{1}{2}\sigma}}{w}$$
$$\times [\gamma^2 I_4(\tilde{\eta}|\eta) + (Q_0 k_3/k_1)^2 I_2(\tilde{\eta}|\eta)] + O(w^{-2}), \quad \text{(B 2)}$$

where
$$I_n(\tilde{\eta}|\eta) = \frac{1}{2}\int_{\eta}^{\tilde{\eta}} \frac{1}{\hat{\eta}^n}\frac{d\hat{\eta}}{(1-\hat{\eta}^2)^{\sigma}}. \quad \text{(B 3)}$$

Appendix C

In this appendix we expand (3.42) in a power series in r. We do this by using the fact (Carrier, Krook & Pearson 1966, p. 356) that, if $F(s)$ has the expansion

$$F(s) = \frac{1}{s}\sum_{n=1}^{\infty} B_n s^{-n}, \quad \text{(C 1)}$$

where the B_n are independent of s, its Laplace transform will have the expansion

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\, e^{sr}\, ds = \sum_{n=1}^{\infty} \frac{B_n}{n!} r^n. \quad \text{(C 2)}$$

It can be seen from the expansion (B 2) for W that it will be more convenient to work in terms of the variable $w/\eta^{\frac{3}{2}}(1-\eta^2)^{\frac{1}{2}\sigma}$ than to work in terms of s. Thus, since (3.1), (3.21) and (3.30) imply that

$$[s - i(U_\infty k_1/U_r)]\, r = iw(k_1 r/\eta Q_0),$$

it follows from (C 1) and (C 2) that if

$$\frac{\exp\left[-\gamma\int_{\tilde{\eta}}^{\eta} \frac{d\hat{\eta}}{\hat{\eta} W(s,\eta|\hat{\eta})}\right]}{W(s,\eta|\tilde{\eta})\,[W(s,\eta|\tilde{\eta}) - \gamma Q_0 \alpha_1(\tilde{\eta}) - k_2\gamma Q_0 \alpha_2(\tilde{\eta})/k_1]^m} \quad \text{(C 3)}$$

has the power-series expansion

$$\frac{1}{[\tilde{\eta}^{\frac{3}{2}}(1-\tilde{\eta}^2)^{\frac{1}{2}\sigma}]^{1+m}} \sum_{n=1}^{\infty} B_n^{(m)} \left[\frac{\eta^{\frac{3}{2}}(1-\eta^2)^{\frac{1}{2}\sigma}}{w}\right]^{n+1} \quad \text{(C 4)}$$

then K_m will have the expansion

$$K_m = \frac{1}{[\tilde{\eta}^{\frac{3}{2}}(1-\tilde{\eta}^2)^{\frac{1}{2}\sigma}]^{m-1}} \sum_{n=0}^{\infty} \frac{B_{n+1}^{(m)}}{(n+1)!}\left[\frac{ik_1 r\eta^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}\sigma}}{Q_0}\right]^n \quad \text{(C 5)}$$

where the B_n are functions of η and $\tilde{\eta}$. In order to obtain the expansion (C 4) we must assume that k_2 and k_3 remain finite as $w \to \infty$. Then substituting (B 2) into (C 3) and expanding the result for large w, we find, after some rearrangement, that

$$\left.\begin{aligned} B_1^{(1)} &= 1,\\ B_2^{(1)} &= B_0(\tilde{\eta}|\eta) - \tfrac{1}{2}\gamma J(\tilde{\eta}|\eta),\\ B_3^{(1)} &= \tfrac{5}{4}\gamma^2[J(\tilde{\eta}|\eta)]^2 + [B_0(\tilde{\eta}|\eta)]^2 - 2[\gamma^2 I_4(\tilde{\eta}|\eta) + (k_3 Q_0/k_1)^2 I_2(\tilde{\eta}|\eta)],\\ &\cdot\\ &\cdot\\ &\cdot\\ &\cdot \end{aligned}\right\} \tag{C 6}$$

$$\left.\begin{aligned} B_1^{(2)} &= 0,\\ B_2^{(2)} &= 1,\\ B_3^{(2)} &= 2B_0(\tilde{\eta}|\eta),\\ &\cdot\\ &\cdot\\ &\cdot\\ &\cdot \end{aligned}\right\} \tag{C 7}$$

where
$$B_0(\tilde{\eta}|\eta) = \frac{Q_0}{k_1}\frac{\mu(\tilde{\eta})}{\tilde{\eta}^{\frac{1}{2}}(1-\tilde{\eta}^2)^{\frac{1}{2}\sigma}} - \frac{5\gamma}{2}J(\tilde{\eta}|\eta). \tag{C 8}$$

We can evaluate the integral in (3.42) for large values of w even if k_2/k_1 is allowed to be $O(w)$. We must then account for the poles in the integrand and the result is rather complicated. It can be written as

$$\begin{aligned} rK_1 = \int_0^r \Bigg\{ & e^{i\lambda\tilde{r}} + \frac{\gamma i k_1 \hat{r}(\eta)}{Q_0} J\left(\frac{\tilde{r}}{2}e^{i\lambda\tilde{r}} - \int_0^{\tilde{r}} e^{i\lambda\hat{r}}\,d\hat{r}\right) - \frac{k_1^2 \hat{r}^2(\eta)}{Q_0^2}\\ & \times \left[\frac{\gamma^2 J^2}{2}\left(\frac{\tilde{r}^2}{4}e^{i\lambda\tilde{r}} - \int_0^{\tilde{r}} \hat{r}e^{i\lambda\hat{r}}\,d\hat{r}\right) + \tfrac{3}{2}\gamma^2 J^2 \int_0^{\tilde{r}}\int_0^{\hat{r}} e^{i\hat{\hat{r}}}\,d\hat{\hat{r}}\,d\hat{r}\,d\hat{r}\right.\\ & \left.\left. - (\gamma^2 I_4 + (k_3 Q_0/k_1)^2 I_2)\,\tilde{r}\int_0^{\tilde{r}} e^{i\lambda\hat{r}}\,d\hat{r}\right]\right\} d\tilde{r} + O((k_1 r)^3), \end{aligned} \tag{C 9}$$

$$rK_2 = \frac{ik_1}{Q_0\tilde{\eta}}\frac{\hat{r}(\eta)}{\hat{r}(\tilde{\eta})}\int_0^r \left\{\tilde{r}e^{i\lambda\tilde{r}} + \frac{i\gamma J k_1}{Q_0}\hat{r}(\eta)\left[\frac{\tilde{r}^2}{2}e^{i\lambda\tilde{r}} - \int_0^{\tilde{r}} \hat{r}e^{i\lambda\hat{r}}\,d\hat{r}\right]\right\} d\tilde{r} + O((k_1 r)^3), \tag{C 10}$$

where
$$\lambda(\tilde{\eta}|\eta) = (k_1/Q_0)\,B_0(\tilde{\eta}|\eta)\,\hat{r}(\eta), \quad \hat{r}(\eta) \equiv \eta^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}\sigma} \qquad \text{(C 11), (C 12)}$$

and in order to simplify the writing we have omitted the arguments of $J(\tilde{\eta}|\eta)$, $\lambda(\tilde{\eta}|\eta)$ and $I_n(\tilde{\eta}|\eta)$.

The indicated integrations can be carried out explicitly but for simplicity in writing they are left unintegrated. The result is valid for small values of $k_1 r$ and arbitrary values of $k_2 r$.

Appendix D

In order to obtain a relation between the upstream vorticity vector

$$\boldsymbol{\omega}_\infty(\mathbf{x}-\hat{\mathbf{\imath}}U_\infty t) \equiv \nabla \times \mathbf{u}_\infty(\mathbf{x}-\hat{\mathbf{\imath}}U_\infty t) \tag{D 1}$$

and the curl of the first term on the right side of (2.33), we use the chain rule and the fact that the permutation tensor $\epsilon_{i,j,k}$ possesses the odd symmetry $\epsilon_{i,j,k} = -\epsilon_{i,k,j}$ to obtain

$$\begin{aligned}\omega_i^{(1)} &\equiv \epsilon_{i,j,k}\frac{\partial}{\partial x_j}\mathbf{u}_\infty(\mathbf{X}-\hat{\mathbf{\imath}}U_\infty t)\cdot\frac{\partial \mathbf{X}}{\partial x_k}\\ &= \epsilon_{i,j,k}\frac{\partial u_{\infty n}}{\partial X_m}\frac{\partial X_m}{\partial x_j}\frac{\partial X_n}{\partial x_k} = \tfrac{1}{2}\epsilon_{i,j,k}\frac{\partial u_{\infty n}}{\partial X_m}(\delta_{n,s}\delta_{m,t}-\delta_{n,t}\delta_{m,s})\frac{\partial X_s}{\partial x_k}\frac{\partial X_t}{\partial x_j}.\end{aligned} \tag{D 2}$$

Then since $\epsilon_{r,s,t}\epsilon_{r,n,m} = \delta_{n,s}\delta_{m,t}-\delta_{n,t}\delta_{m,s}$, this becomes

$$\omega_i^{(1)} = \omega_{\infty r}(\mathbf{X}-\hat{\mathbf{\imath}}U_\infty t)\,\lambda_{r,i},$$

where we have put
$$\lambda_{r,i} = -\tfrac{1}{2}\epsilon_{i,j,k}\epsilon_{r,s,t}\frac{\partial X_s}{\partial x_k}\frac{\partial X_t}{\partial x_j}$$

and as usual $\boldsymbol{\omega}_\infty(\mathbf{X}-\hat{\mathbf{\imath}}U_\infty t)$ denotes the result obtained by replacing $\mathbf{x}$ with $\mathbf{X}$ in (D 1). But since the Laplace development of the determinant

$$\left|\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right| \equiv \begin{vmatrix}\partial X_1/\partial x_1 & \partial X_2/\partial x_1 & \partial X_3/\partial x_1\\ \partial X_1/\partial x_2 & \partial X_2/\partial x_2 & \partial X_3/\partial x_2\\ \partial X_1/\partial x_3 & \partial X_2/\partial x_3 & \partial X_3/\partial x_3\end{vmatrix}$$

implies that
$$\epsilon_{n,k,j}\left|\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right| = \epsilon_{r,s,t}\frac{\partial X_s}{\partial x_k}\frac{\partial X_t}{\partial x_j}\frac{\partial X_r}{\partial x_n},$$

it follows that
$$\lambda_{r,i}\,\partial X_r/\partial x_n = |\partial \mathbf{X}/\partial \mathbf{x}|\,\delta_{i,n}$$

and therefore that the matrix whose elements are the $\lambda_{r,i}$ divided by $|d\mathbf{X}/\partial \mathbf{x}|$ is the inverse of the matrix $[\partial X_r/\partial x_i]$. On the other hand, it follows from the identity

$$\frac{\partial X_r}{\partial x_k}\frac{\partial x_k}{\partial a_i} = \frac{\partial a_r}{\partial x_k}\frac{\partial x_k}{\partial a_i} = \delta_{r,i},$$

where the $a_i \equiv X_i - \delta_{i,1}U_\infty t$ are essentially Lagrangian co-ordinates, that $[\partial x_k/\partial a_i]$ is also the inverse of $[\partial X_r/\partial x_i]$. Then since this inverse is certainly unique, it follows that

$$\lambda_{r,i} = \left|\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right|\frac{\partial x_r}{\partial a_i} = \left|\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right|\frac{\partial x_r}{\partial a_i}$$

and therefore that
$$\omega_i^{(1)} = \left|\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right|\boldsymbol{\omega}_\infty(\mathbf{X}-\hat{\mathbf{\imath}}U_\infty t)\cdot\frac{\partial \mathbf{x}}{\partial a_i}.$$

Appendix E. List of commonly used symbols

c_0	speed of sound of mean flow
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
$\hat{\mathbf{\imath}}$	unit vector in x_1 co-ordinate direction

$\mathbf{k} = (k_1, k_2, k_3)$	wavenumber vector
$\hat{\mathbf{n}}$	unit normal to surface of obstacle
p	pressure
p'	pressure fluctuation
p_0	mean pressure
s'	entropy fluctuation
s_∞	imposed upstream entropy distortion
S	entropy
t	time
$\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$	velocity perturbation
$\mathbf{U} = (U_1, U_2, U_3) = (U_x, U_y, U_z)$	potential flow velocity (steady)
U_∞	constant mean-flow velocity at upstream infinity
$\mathbf{u}_\infty$	upstream distortion velocity
$\mathbf{u}^{(I)}$	incident distortion velocity given by (2.33)
$\mathbf{v}$	fluid velocity
$\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$	position vector
$\mathbf{X} = (X_1, X_2, X_3) = (U_\infty \Delta, Y, Z)$	vector composed of integrals of equations for streamlines
Y, Z	integrals of the system (2.20) characterized by (2.21)
γ	defined by (3.7)
Δ	drift function defined by (2.22)
$\kappa = c_p/c_v$	specific-heat ratio
ρ	density
ρ_0	mean-flow density
ρ'	density fluctuation
ρ_∞	mean density at upstream infinity
σ	$1/2\gamma^2$
ϕ	perturbation potential
Φ	velocity potential for mean potential flow
Ψ	stream function for mean potential flow
$\boldsymbol{\omega} = \{\omega_1, \omega_2, \omega_3\}$	unsteady vorticity vector $\nabla \times \mathbf{u}$

Superscripts:

prime	fluctuating quantities

Subscripts:

0	mean-flow conditions
∞	upstream conditions

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